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Perturbations of Dirac operators

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Abstract

We study general conditions under which the computations of the index of a perturbed Dirac operator $D_s = D + sZ$ localize to the singular set of the bundle endomorphism Z in the semiclassical limit $s \to \infty$. We show how to use Witten's method to compute the index of D by doing a combinatorial computation involving local data at the nondegenerate singular points of the operator Z. In particular, we provide examples of novel deformations of the de Rham operator to establish new results relating the Euler characteristic of a spin^c manifold to maps between its even and odd spinor bundles.

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1. Introduction

About 20 years ago Witten [18] introduced a beautiful new approach to proving Morse inequalities based on the deformation of the de Rham complex. His ideas were fruitfully applied in many specific situations. The purpose of this paper is to study general conditions under which one can use the method of Witten deformations to obtain an expression for the index of any Dirac-type operator in terms of local quantities associated to singular sets of bundle maps. The operators and bundle maps considered in this paper are more general and include those considered by other researchers as special cases.

We now describe the setup of our paper. See Appendix B for a review of graded Clifford bundles and Dirac operators. Let $E = E^+ \oplus E^-$ be a graded self-adjoint Clifford module over a closed, smooth, Riemannian manifold M. Let $\Gamma(M, E)$ denote the space of smooth sections of E and $D : \Gamma(M, E) \to \Gamma(M, E)$ be the Dirac operator associated to a Clifford module E.

Let $D^{\pm} : \Gamma\left(M, E^{\pm}\right) \to \Gamma\left(M, E^{\mp}\right)$ denote the restrictions of the Dirac operator to smooth even or odd sections. Observe that $D^- = \left(D^+\right)^*$, the L^2-adjoint of D^+. Let $Z^+ : \Gamma\left(M, E^+\right) \to \Gamma\left(M, E^-\right)$ be a smooth bundle map, and we let Z^- denote the adjoint of Z^+. The operator Z on $\Gamma(M, E)$, defined by $Z\left(v^+ + v^-\right) = Z^- v^- + Z^+ v^+$ for any $v^+ \in E_x^+$ and $v^- \in E_x^-$, is self-adjoint. A *generalized Witten deformation* of D is a family D_s of perturbed

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differential operators

$$D_s = (D + sZ) : \Gamma(M, E) \to \Gamma(M, E).$$

We define the operators $D_s^{\pm}$ by restricting in the obvious ways. Our definition includes the known examples of Witten deformation as special cases.

It is well-known [2] that the index $\operatorname{ind}\left(D^+\right)$ of D^+ depends only on the homotopy type of the principal symbol and satisfies

$$\operatorname{ind}(D) = \dim\ker\left((D_s)^2\Big|_{\Gamma(M,E^+)}\right) - \dim\ker\left((D_s)^2\Big|_{\Gamma(M,E^-)}\right).$$

Thus, we need to study the operator

$$(D_s)^2 = D^2 + s(ZD + DZ) + s^2 Z^2.$$

The leading order behavior of the eigenvalues of this operator as $s \to \infty$ is determined by combinatorial data at the singular set of the operator Z. This "localization" allows one to compute $\operatorname{ind}(D)$ in terms of that data.

The results of this paper hold if the perturbation Z is a *proper perturbation*. That is, it satisfies two conditions:

(1) $(D_s)^2 - (D)^2 = s(ZD + DZ) + s^2 Z^2$ is a bundle map.
(2) Each singular point $\bar{x}$ of Z is a *proper singular point*; that is,
 (a) $Z(\bar{x}) = 0$.
 (b) In local coordinates x on a sufficiently small neighborhood U of $\bar{x}$, there exists a constant $c > 0$ such that for all $\alpha \in \Gamma(U, E)$ and all $x \in U$,
 $$\|Z\alpha\|_x \geq c\,|x - \bar{x}|\,\|\alpha\|_x,$$
 where $\|\cdot\|_x$ is the pointwise norm on E_x.

We should note that in the important case when Z is Clifford multiplication by a vector field, Condition (1) is not satisfied, and localization typically does not occur (see Appendix A).[1] In the proof of the Poincaré–Hopf index theorem, $Z = V^\flat \wedge + V \lrcorner$, and Condition (2) reduces to the requirement that the vector field has nondegenerate zeros.

In Section 2, we classify the possible gradings of E compatible with the existence of such Z. We also establish necessary and sufficient conditions on the dimension of E and on the form of the operator Z in order for (1) and (2) to be satisfied.

In Section 3, we show that conditions (1) and (2) imply that the singular set of Z consists of a finite number of nondegenerate zeros, and as $s \to \infty$ the bounded spectrum of the Witten Laplacian localizes to the singular set of Z. This means that the index of D can be computed by studying the zero spectrum of limiting "model" operators, which turn out to be harmonic oscillators. The main tool is the localization theorem of Shubin [16].

If a proper perturbation with no singularities exists, then the index is zero. In particular, let $E^{\pm} \cong (\mathbb{S} \otimes W)^{\pm} = \left(\mathbb{S}^+ \otimes W^{\pm}\right) \oplus \left(\mathbb{S}^- \otimes W^{\mp}\right)$ be any graded, self-adjoint Clifford module over an even-dimensional, spin^c manifold M (all such Clifford modules have this form; see Corollary B.2 in Appendix B). Then the index of the Dirac operator corresponding to this Clifford module is zero if the bundles W^+ and W^- are isomorphic (see Corollary 3.5).

In Section 4, we consider an elliptic operator of the form $Q = \sum\left(A_j \partial_j + x_j B_j\right)$ on $\mathbb{C}^m$-valued functions on $\mathbb{R}^n$, such that each A_j and B_k is an $m \times m$ matrix and $\sum x_k B_k$ is a proper perturbation. We prove that Q is Fredholm and that continuous families of such operators have the same index.

The main results of the paper are Theorems 5.1 and 5.4, which express the index of the Dirac operator D in terms of the local information at the singular points. We show in Theorem 5.1 that the index of the Dirac operator D is the sum of indices of operators on vector-valued functions on $\mathbb{R}^n$ as in Section 4, where the coefficients A_j and B_k depend only on local data at each singular point $\overline{x}$. Assuming typical properties of Z near each $\overline{x}$, the indices may be computed more explicitly, as shown in Theorem 5.4. We emphasize strongly that all the information necessary to compute the index of D is contained in the set of matrices of first derivatives of Z and in the Clifford matrices taken at each singular point $\overline{x}$. Thus this information is local in nature, and the answer is easily obtainable.

[1] If the vector field is a generator of an action by a one-parameter group of isometries, the localization occurs when the Witten Laplacian is restricted to each eigenspace of the Lie derivative associated to this generator. We plan to treat this situation in a paper currently under preparation.

In Section 6 we apply our results to the geometric Dirac operators. In particular, we use Corollary 5.6 to obtain the Poincaré–Hopf theorem. Our proof also yields a new result, that the Euler characteristic of an even-dimensional, spin^c manifold is zero if and only if the even and odd spin^c bundles are isomorphic (see Corollary 6.7). In Theorem 6.11, we show that the Euler characteristic of a spin^c manifold is the sum of the indices of zeros of a possibly singular section of the conformal pin bundle over the manifold. Thus, the Euler characteristic is zero if and only if the odd pin bundle $\mathrm{Pin}^-(T^*M)$ has a global section. In Section 6.3, we use our results to show that if M is a submanifold of odd codimension in a manifold endowed with a graded Clifford module, then the index of the Dirac operator associated to the restriction of this Clifford bundle to M is zero.

Witten deformation was first introduced in [18], where the author sketched a beautiful proof of the Morse inequalities by deforming the de Rham complex (see also [7,16,15,14]). In [8] B. Helffer and J. Sjöstrand put Witten's analysis on a rigorous footing. In addition Witten suggested a way to use his method to prove the Poincaré–Hopf theorem. Rigorous treatments of his ideas in this direction are contained in [17,19].

2. Perturbing Dirac operators

2.1. Preliminaries and notational conventions

Throughout this paper, the manifold M is always assumed to be a smooth, closed, oriented Riemannian manifold of dimension n, and $E = E^+ \oplus E^-$ is assumed to be a graded, self-adjoint, Hermitian Clifford module over M. If M is spin^c, then $\mathbb{S}$ always denotes a complex spinor bundle over M, a particular example of such a bundle E. We define $D : \Gamma(M, E) \to \Gamma(M, E)$ to be the corresponding Dirac operator, and let $D^\pm : \Gamma\left(M, E^\pm\right) \to \Gamma\left(M, E^\mp\right)$ denote the restrictions of D to smooth even and odd sections. The operator D^- is the adjoint of D^+ with respect to the L^2-metric on $\Gamma(M, E)$ defined by the Riemannian metric on M and the Hermitian metric on E.

In the following, we denote Clifford multiplication by an element $v \in T_xM$ on the fiber E_x by $c(v)$. Clifford multiplication by cotangent vectors will use the same notation: $c(\alpha) := c\left(\alpha^\#\right)$, where $T_x^*M \overset{\#}{\to} T_xM$ is the metric isomorphism.

The natural grading on E is induced by the action of the chirality operator γ. Recall that if $e_1, \ldots, e_n$ is an oriented orthonormal basis of T_xM, then the chirality operator is multiplication by

$$\gamma = i^k c(e_1) \ldots c(e_n) \in \mathrm{End}(E_x),$$

where $k = n/2$ if n is even and $k = (n+1)/2$ if n is odd. In this paper we study the other possible gradings as well. See the appendix (Appendix B) for more information.

Let $Z^+ \in \Gamma\left(M, \mathrm{Hom}\left(E^+, E^-\right)\right)$ be a smooth bundle map, and let Z^- denote the adjoint of Z^+. The operator Z on $\Gamma(M, E)$, defined by $Z\left(v^+ + v^-\right) = Z^-v^- + Z^+v^+$ for any $v^+ \in E_x^+$ and $v^- \in E_x^-$, is self-adjoint. Let D_s denote the perturbed Dirac operator

$$D_s = (D + sZ) : \Gamma(M, E) \to \Gamma(M, E), \tag{2.1}$$

and define the operators $D_s^\pm$ by restricting in the obvious ways.

2.2. Nonexistence of perturbations compatible with the natural grading

For any differential operator L, let σ_L denote its principal symbol. We will start with the result that holds for general first-order operators.

Lemma 2.1. *Let $L : \Gamma(M, E) \to \Gamma(M, E)$ be a smooth, first-order differential operator, and let $Z : \Gamma(M, E) \to \Gamma(M, E)$ be a bundle endomorphism. Then the operator $LZ + ZL$ is a bundle map if and only if $Z \circ \sigma_L(x, \xi) + \sigma_L(x, \xi) \circ Z = 0$ on E_x for every $x \in M$, $\xi \in T_x^*M$.*

Proof. The differential operator $ZL + LZ$ is zeroth order if and only if it commutes with multiplication m_f by any smooth function f on M. We calculate the commutator

$$\begin{aligned}\left[ZL + LZ, m_f\right] &= Z\left[L, m_f\right] + \left[L, m_f\right] Z \text{ since } Z \text{ is zeroth order}\\ &= \mathrm{i}(Z \circ \sigma_L(df) + \sigma(L)(df) \circ Z),\end{aligned}$$

where for any 1-form α on M, $\sigma_L(\alpha)$ is the bundle endomorphism defined by $\sigma_L(\alpha)|_x = \sigma_L(x, \alpha_x)$. □

Since

$$(D_s)^2 - D^2 = s(ZD + DZ) + s^2 Z^2, \tag{2.2}$$

we have the following corollary:

Corollary 2.2. *For any $s \neq 0$ the operator $(D_s)^2 - D^2$ is zeroth order if and only if $Z \circ \sigma_D(x, \xi) + \sigma_D(x, \xi) \circ Z = 0$ on E_x for every $x \in M$, $\xi \in T_x^* M$.*

Remark 2.3. Corollary 2.2 is true in even greater generality, such as when D is a first-order, classical pseudodifferential operator. However, in this paper we consider only differential operators.

A bundle endomorphism Z satisfying the condition in Corollary 2.2 does not always exist. In particular, the following result applies to the spin^c Dirac operator, whose principal symbol is $ic(\xi)$ (for $\xi \in T^*M$).

Proposition 2.4. *Let V be an even-dimensional, oriented, Euclidean vector space. Let $\mathbb{S} = \mathbb{S}^+ \oplus \mathbb{S}^-$ denote the associated space of complex spinors. There does not exist a linear map $Z : \mathbb{S} \to \mathbb{S}$ such that $Z \circ c(v) + c(v) \circ Z = 0$ for every $v \in V$ and such that Z restricts to a map $Z : \mathbb{S}^+ \to \mathbb{S}^-$.*

Proof. Any endomorphism of $\mathbb{S}$ can be written as Clifford multiplication by an element of $\mathbb{C}\mathrm{l}(V)$, so that the result is equivalent to the statement that no element of $\mathbb{C}\mathrm{l}(V)$ anticommutes with every vector. This is a consequence of the elementary fact that if $\alpha \in \mathbb{C}\mathrm{l}(V)$ anticommutes with every vector, then α is a complex multiple of the chirality operator. Since this element maps $\mathbb{S}^+$ to itself, the result follows. □

The corollary below is a generalization.

Corollary 2.5. *Let V be an even-dimensional, oriented, Euclidean vector space. Let $\mathbb{S} = \mathbb{S}^+ \oplus \mathbb{S}^-$ denote the associated space of complex spinors. Let W be another vector space, and consider the Clifford action $\widetilde{c}(v) = c(v) \otimes \mathbf{1}$ on $\mathbb{S} \otimes W$. There does not exist a linear map $Z : \mathbb{S} \otimes W \to \mathbb{S} \otimes W$ such that $Z \circ \widetilde{c}(v) + \widetilde{c}(v) \circ Z = 0$ for every $v \in V$ and such that Z restricts to a map $Z : \mathbb{S}^+ \otimes W \to \mathbb{S}^- \otimes W$.*

Proof. Endow W with a Euclidean (or Hermitian) metric. Choose an orthonormal basis $\{w_1, \ldots, w_k\}$ of W. Write

$$Z(v \otimes w_i) = \sum_j Z_{ij}(v) \otimes w_j;$$

the corollary follows from the proposition applied to each linear operator Z_{ij} separately. □

Example 2.6. The Dolbeault and signature operators do not have such perturbations, because in both cases the Clifford action has the form of Corollary 2.5.

2.3. Admissible perturbations

The following results determine the precise form of perturbations Z satisfying the condition in Corollary 2.2 if D is the Dirac operator associated to a Clifford bundle over a spin^c manifold.

Proposition 2.7. *Let M be even-dimensional and* spin^c. *Let $E^\pm \cong (\mathbb{S} \otimes W)^\pm = \left(\mathbb{S}^+ \otimes W^\pm\right) \oplus \left(\mathbb{S}^- \otimes W^\mp\right)$ be any Clifford module over M (see Corollary B.2 in the appendix). Suppose that there is a bundle endomorphism $Z^+ : \Gamma\left(M, (\mathbb{S} \otimes W)^+\right) \to \Gamma\left(M, (\mathbb{S} \otimes W)^-\right)$ such that the self-adjoint operator $Z = \left(Z^+, \left(Z^+\right)^*\right) : \mathbb{S} \otimes W \to \mathbb{S} \otimes W$ anticommutes with Clifford multiplication by vectors. Then Z has the form $Z = \gamma \otimes \phi$, where γ is the chirality operator on $\mathbb{S}$ and where $\phi^+ : W^+ \to W^-$ is a bundle map with $\phi = \left(\phi^+, \left(\phi^+\right)^*\right)$. Conversely, any bundle endomorphism of that form anticommutes with Clifford multiplication by vectors.*

Proof. The action of Z on $\mathbb{S} \otimes W$ has the following local form. For a local orthonormal basis $\{b_1, \ldots, b_k\}$ of W and any $\alpha \in \Gamma(M, \mathbb{S})$,

$$Z(\alpha \otimes b_i) = \sum_j Z_{ij}(\alpha) \otimes b_j,$$

where each operator Z_{ij} must anticommute with Clifford multiplication by vectors. Thus $Z_{ij} = c_{ij}\,\gamma(\alpha)$ for some complex scalar c_{ij} and

$$Z(\alpha \otimes b_i) = \sum_j c_{ij}\gamma(\alpha) \otimes b_j = \gamma(s) \otimes \sum_j c_{ij}\, b_j.$$

The operator γ restricts to $\mathbf{1}$ on $\mathbb{S}^+$ and $-\mathbf{1}$ on $\mathbb{S}^-$. The result follows from the hypothesis and the equation above. □

Remark 2.8. The case when W^- is zero-dimensional is reflected in Corollary 2.5, where the only possible bundle map ϕ is the zero map.

Remark 2.9. The restriction of such a ϕ to a fiber is invertible if and only if $\dim W^+ = \dim W^-$.

Proposition 2.10. *Let M be odd-dimensional and* spin^c. *Fix the representation c^+ of $\mathbb{C}l(TM)$. Let $E \cong \mathbb{S} \otimes (W' \oplus W')$ be as in Corollary B.2. Suppose that there exists a self-adjoint endomorphism $Z = \left(Z^+, \left(Z^+\right)^*\right) : E \to E$ that anticommutes with the Clifford multiplication $\left(c^+(v) \otimes \mathbf{1},\ c^-(v) \otimes \mathbf{1}\right)$ by all vectors $v \in TM$. Then Z has the form $Z = \mathbf{1} \otimes \begin{pmatrix} 0 & \phi \\ -\phi & 0 \end{pmatrix}$, where $\phi : W' \to W'$ is a skew-adjoint bundle map. Conversely, any bundle endomorphism of that form is self-adjoint and anticommutes with Clifford multiplication by vectors.*

Proof. Since $\mathbb{C}l^+ \cong \operatorname{End}(\mathbb{S})$ in odd dimensions, no nonzero element of $\operatorname{End}(\mathbb{S})$ anticommutes with Clifford multiplication c^+ by all vectors. An endomorphism Z that anticommutes with

$$\begin{aligned} c^E(v) &= \left(c^+(v) \otimes \mathbf{1}, c^-(v) \otimes \mathbf{1}\right) \\ &= c^+(v) \otimes \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix} \end{aligned}$$

for all vectors v must have the following properties. First, $Z : \mathbb{S} \otimes W \to \mathbb{S} \otimes W$ must commute with all maps of the form $A \otimes \mathbf{I}$ with $A \in \operatorname{End}(\mathbb{S})$, because any such map can be realized as multiplication by an element in $\mathbb{C}l^+$ in odd dimensions. It follows easily that Z can be expressed in the form $Z = \mathbf{1} \otimes Z'$, where $Z' = \begin{pmatrix} Z_1 & Z_2 \\ Z_3 & Z_4 \end{pmatrix}$is an endomorphism of $W' \oplus W'$. Because Z' maps span $\left\{(w, \pm w) \in W' \oplus W' \mid w \in W'\right\}$ to span $\left\{(w, \mp w) \in W' \oplus W' \mid w \in W'\right\}$, $Z_1 \pm Z_2 = \mp Z_3 - Z_4$. These equations imply that $Z_4 = -Z_1$ and $Z_3 = -Z_2$. Because Z anticommutes with $c^E(v)$, $Z_1 = Z_4 = 0$. Self-adjointness of Z implies further that Z_2 is skew-adjoint, and the result follows. □

Remark 2.11. If M is not spin^c, Propositions 2.7 and 2.10 above remain true locally.

2.4. Proper perturbations of Dirac operators

In this section, we state the nondegeneracy conditions on the perturbation. In previously studied types of perturbations, our conditions are equivalent to those required by others (see the introduction).

Definition 2.12. Let $Z : E \to E$ be a smooth bundle map. We say that $\overline{x} \in M$ is a **proper singular point** of Z if on a sufficiently small neighborhood U of $\overline{x}$ we have

(1) $Z|_{\overline{x}} = 0$, and
(2) in local coordinates x on U, there exists a constant $c > 0$ such that for all $\alpha \in \Gamma(U, E)$,

$$\|Z\alpha\|_x \geq c\,|x - \overline{x}|\,\|\alpha\|_x,$$

where $\|\cdot\|_x$ is the pointwise norm on E_x.

Lemma 2.13. *A point $\overline{x} \in M$ is a proper singular point of Z if and only if, in local coordinates x on U, there exist invertible bundle maps Z_j for $1 \leq j \leq n = \dim M$ over U such that $Z = \sum_j (x - \overline{x})_j Z_j$ on U, and Z is invertible over $U \setminus \{\overline{x}\}$.*

Proof. Since Z is smooth and vanishes at $\overline{x}$, $Z=\sum_j (x-\overline{x})_j Z_j$ for some bundle maps Z_j. The inequality in the definition above is equivalent to

$$\left\|\sum_j \sigma_j Z_j \alpha\right\|_x^2 \geq c^2$$

for every $\sigma \in S^{n-1}$, $\alpha \in \Gamma\left(\overline{U}, E\right)$, $x \in U$ such that $\|\alpha\|_x = 1$. Since the left hand side of the inequality is a continuous function of σ and α over the compact set $S^{n-1} \times \left\{\alpha \in \Gamma\left(\overline{U}, E\right) \mid \|\alpha\|_x = 1 \text{ for all } x \in U\right\}$, its infimum is attained. It follows that Z is invertible away from $\overline{x}$ if and only if the inequality holds. □

Definition 2.14. Let $D^{\pm} : \Gamma\left(M, E^{\pm}\right) \to \Gamma\left(M, E^{\mp}\right)$ be the Dirac operator associated to a bundle of graded Clifford modules. Let $D_s = D + sZ$ for $s \in \mathbb{R}$, where $Z = \left(Z^+, \left(Z^+\right)^*\right) \in \Gamma\left(M, \mathrm{End}\left(E^+, E^-\right)\right)$. We say that Z is a **proper perturbation of** D if

(1) $(D_s)^2 - D^2$ is a zeroth order operator.
(2) All singular points of Z are proper.

Remark 2.15. If M is compact, then the number of singular points of a proper perturbation is finite.

The following lemma will be used to quantify ranks of vector bundles on which the proper perturbations act.

Lemma 2.16. *There exists a linear map $L : \mathbb{R}^k \to M_r(\mathbb{C})$ that satisfies $L(x) \in \mathrm{Gl}(r, \mathbb{C})$ for $x \neq 0$ if and only if $r = m2^{\left\lfloor \frac{k-1}{2} \right\rfloor}$ for some positive integer m.*

Proof. Since $L(x)$ could be Clifford multiplication by the vector x on the r-dimensional vector space $\mathbb{S}\otimes\mathbb{C}^m$, ranks of the form $r = m2^{\left\lfloor \frac{k}{2} \right\rfloor}$ are realizable. If k is even, the image of the restriction of Clifford multiplication to $\mathbb{S}^+ \otimes \mathbb{C}^m$ has rank $r = m2^{\frac{k}{2}-1}$. Hence for all positive integers k and m, there exist linear maps $L : \mathbb{R}^k \to M_r(\mathbb{C})$ with $r = m2^{\left\lfloor \frac{k-1}{2} \right\rfloor}$ and $L(x) \in \mathrm{Gl}(r, \mathbb{C})$ for $x \neq 0$.

[2]Next, suppose that a linear map $L : \mathbb{R}^k \to M_r(\mathbb{C})$ satisfies $L(x) \in \mathrm{Gl}(r, \mathbb{C})$ for $x \neq 0$, for some positive integers k and r. Such a map restricts to a map $L : S^{k-1} \to \mathrm{Gl}(r, \mathbb{C})$ with $L(-x) = -L(x)$. Consider the vector bundles $r\mathbf{T} = \mathbb{R}P^{k-1} \times \mathbb{C}^r$ and rL_k (r times the complexification of the canonical line bundle) over the projective space $\mathbb{R}P^{k-1}$. Note that

$$\begin{aligned}&\mathbb{R}P^{k-1} \times \mathbb{C}^r = S^{k-1} \times \mathbb{C}^r / (x, y) \sim (-x, y), \quad \text{and}\\ &rL_k = S^{k-1} \times \mathbb{C}^r / (x, y) \sim (-x, -y).\end{aligned}$$

The map $f : S^{k-1} \times \mathbb{C}^r \to S^{k-1} \times \mathbb{C}^r$ defined by $f(x, y) = (x, L(x)y)$ induces an isomorphism between rL_k and $r\mathbf{T}$. Thus, the virtual bundle $r(L_k - \mathbf{T})$ represents the zero element in the reduced complex K-group $\widetilde{K}\left(\mathbb{R}P^{k-1}\right)$. Since

$$\widetilde{K}\left(\mathbb{R}P^{k-1}\right) \cong \mathbb{Z}_{2^{\left\lfloor \frac{k-1}{2} \right\rfloor}}$$

with generator $L_k - \mathbf{T}$ (see [1]), we must have $r = m2^{\left\lfloor \frac{k-1}{2} \right\rfloor}$ for some positive integer m. □

The following two theorems give necessary and sufficient conditions on the bundle E and the bundle map Z in order that Z be a proper perturbation.

Theorem 2.17. *Suppose that the dimension n of M is even. Let Z be a proper perturbation of D on $\Gamma(M, E)$, with notation as in Definition 2.14. Let $\{\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_N\}$ be the singular points of Z, and let U_l be the neighborhood of $\overline{x}_l$ as in the definition. Over each U_l, choose a local* spin^c *bundle and isomorphism $E \cong \mathbb{S} \otimes W$ as in Proposition 2.7. Then the following conditions must be satisfied.*

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[2] This part of the argument is provided in [12,13].

(1) *The local bundles W^+ and W^- from Proposition 2.7 must have the same dimension, which implies that the rank of E must be a multiple of $2^{\frac{n}{2}+1}$. If the set of singular points is nonempty, then the rank must be of the form $m2^n$, where m is a positive integer.*

(2) *Near each singular point $\overline{x}$, the bundle map Z has the form $Z = \sum_j (x-\overline{x})_j \gamma \otimes \phi_j$, with notation as in Definition 2.14 and Proposition 2.7, where each $\phi_j^+ : W^+ \to W^-$ is a locally defined bundle isomorphism with $\phi_j = \left(\phi_j^+, \left(\phi_j^+\right)^*\right)$.*

Conversely, every graded, self-adjoint bundle map Z that has a finite set of singular points and satisfies the two conditions above is a proper perturbation.

Proof. The first part of the first condition and the second condition follow directly from Proposition 2.7. To prove the first condition in the case where the set of singular points is nonempty, suppose that $\overline{x}$ is a singular point of Z, and choose coordinates centered at $\overline{x}$. Identify $W^+_{\overline{x}} \cong W^-_{\overline{x}} \cong \mathbb{C}^d$ for the appropriate positive integer d. Define $L(x) = \sum_{j=1}^n x_j \phi_j^+(\overline{x}) : \mathbb{C}^d \to \mathbb{C}^d$; the hypotheses imply that $L(x)$ is invertible for each $x \neq 0$. Thus,

$$d = m2^{\left\lfloor \frac{n-1}{2} \right\rfloor} = m2^{\frac{n}{2}-1}$$

for some positive integer m, by Lemma 2.16. Since the dimension of $\mathbb{S}$ is $2^{\frac{n}{2}}$, the statement follows. The converse is clear. □

Remark 2.18. The given minimal rank $2^{\frac{n}{2}+1}$ is sharp, since the example given in Proposition 6.14 has precisely that rank.

Theorem 2.19. *Suppose that the dimension n of M is odd. Let Z be a proper perturbation of D on $\Gamma(M, E)$, with notation as in Definition 2.14. Let $\{\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_N\}$ be the singular points of Z, and let U_l be the neighborhood of $\overline{x}_l$ as in the definition. Over each U_l, choose a local* spin^c *bundle and isomorphism $E \cong \mathbb{S} \otimes \left(W' \oplus W'\right)$ as in Proposition 2.10. Then the following condition must be satisfied.*

(1) *If the set of singular points is empty, there is no further restriction on the rank of E; that is, it need only be a multiple of $2^{\frac{(n+1)}{2}}$. If the set of singular points is nonempty, then the rank of E must have the form $m2^n$, where m is a positive integer.*

(2) *Near each singular point, the bundle map Z has the form $Z = \sum_j x_j \mathbf{1} \otimes \begin{pmatrix} 0 & \phi_j \\ -\phi_j & 0 \end{pmatrix}$ as in Proposition 2.10 and in Definition 2.14, where each $\phi_j : W' \to W'$ is a locally defined, skew-adjoint bundle isomorphism.*

Conversely, every graded, self-adjoint bundle map Z that has a finite set of singular points and satisfies the two conditions above is a proper perturbation.

Proof. The second condition follows directly from Proposition 2.10. To prove the first condition in the case where the set of singular points is nonempty, choose local coordinates x centered at a singular point $\overline{x}$. Identify $W'_{\overline{x}} \cong \mathbb{C}^d$ for the appropriate positive integer d. Define $L(x) = \sum_{j=1}^n x_j \phi_j(\overline{x}) : \mathbb{C}^d \to \mathbb{C}^d$; the hypotheses imply that $L(x)$ is invertible for each $x \neq 0$. Thus,

$$d = m2^{\left\lfloor \frac{n-1}{2} \right\rfloor} = m2^{\left(\frac{n-1}{2}\right)}$$

for some positive integer m, by Lemma 2.16. This proves the statement since the dimension of $\mathbb{S}$ is $2^{\frac{n-1}{2}}$, and the converse follows easily. □

The following result shows that nonsingular proper perturbations always exist on Clifford modules over an odd-dimensional manifold.

Proposition 2.20. *Suppose that the dimension n of M is odd. Let E be a bundle of graded Clifford modules over M, and let D be the corresponding Dirac operator. Then there always exists a proper perturbation Z of D; in particular the perturbation may be chosen to be invertible.*

Proof. To prove that proper perturbations always exist, we simply take $\phi = i\mathbf{1}$ in Proposition 2.10. □

Remark 2.21. Note that the perturbation in the proof with $W' = \mathbb{C}$ acts on a bundle of rank $2^{\frac{n+1}{2}}$, so that the minimal rank in Theorem 2.19 is sharp.

3. Localization

In this section we will use [17, Proposition 1.2] to study the asymptotics of the spectrum of D_s^2 as $s \to \infty$. (See also [16].)

Let

$$\begin{aligned} H_s &:= s^{-1}(D_s)^2 = s^{-1}D^2 + ZD + DZ + sZ^2 \\ &= -s^{-1}A + B + sC, \end{aligned}$$

where $-A = D^2$ is a second order, elliptic, self-adjoint operator with a nonnegative principal symbol, and the operators $B = ZD + DZ$ and $C = Z^2$ are self-adjoint bundle maps.

Each of the operators H_s, A, B, and C has two self-adjoint components, acting on $\Gamma\left(M, E^+\right)$ and on $\Gamma\left(M, E^-\right)$, respectively. For example,

$$\begin{aligned} H_s^+ &:= s^{-1}(D_s)^2|_{\Gamma(M,E^+)} = s^{-1}D_s^- D_s^+ : \Gamma\left(M, E^+\right) \to \Gamma\left(M, E^+\right), \quad \text{and} \\ B^- &:= (ZD + DZ)|_{\Gamma(M,E^-)} = Z^+D^- + D^+Z^- : \Gamma\left(M, E^-\right) \to \Gamma\left(M, E^-\right). \end{aligned}$$

We now describe a model operator: a matrix harmonic oscillator that will serve as an approximation of $(D_s)^2$ near $\overline{x}$, a singular point of C. We choose local coordinates $x = (x_1, \ldots, x_n)$ and a trivialization of E near $\overline{x}$. We assume that the volume associated to the Riemannian metric g is the Lebesgue volume element at the point $\overline{x}$ (this is easily done by rescaling if necessary).

In the neighborhood of $\overline{x}$, operator A becomes a $2m \times 2m$ block diagonal differential operator with two $m \times m$ blocks, where $m = \operatorname{rank} E^+ = \operatorname{rank} E^-$. It has the form

$$A = \sum_{1 \le i,j \le n} A_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + A^{(1)}.$$

Note that the operator $A^{(1)}$ is at most first order. Let

$$A^{(2)} = \sum_{1 \le i,j \le n} A_{ij}(\overline{x}) \frac{\partial^2}{\partial x_i \partial x_j};$$

i.e. $A^{(2)}$ is the highest order part of A taken at $x = \overline{x}$, it is a homogeneous second order differential operator with constant coefficients that are $2m \times 2m$ block diagonal Hermitian matrices.

We denote $\overline{B} = B(\overline{x})$, so $\overline{B}$ is just a $2m \times 2m$ block diagonal Hermitian matrix in the chosen trivialization of E.

We also define

$$C^{(2)}(x) = \frac{1}{2} \sum_{1 \le i,j \le n} \frac{\partial^2 C_{ij}}{\partial x_i \partial x_j}(\overline{x})\, x_i x_j,$$

i.e. $C^{(2)}$ is the quadratic part of the potential C near $\overline{x}$.

We define the *model operator* K of H_s at a singular point $\overline{x}$ to be

$$K(\overline{x}) = -A^{(2)} + \overline{B} + C^{(2)}.$$

We denote $m \times m$ blocks of $K(\overline{x})$ as $K^{\pm}(\overline{x})$. Each operator $K^+(\overline{x})$ and $K^-(\overline{x})$ has a discrete spectrum, since it is a quantum Hamiltonian of a k-dimensional harmonic oscillator [16].

Let $\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_N$ be the list of all singular points. Let

$$K^{\pm} = \bigoplus_{l=1}^{N} K^{\pm}(\overline{x}_l)$$

be the model operators for $H_s^{\pm}$ on the set of all singular points of $C = Z^2$. Denote the eigenvalues of $K^{\pm}$ by

$$\mu_1^{\pm} < \mu_2^{\pm} < \mu_3^{\pm} < \cdots$$

and their multiplicities by

$$p_1^{\pm}, p_2^{\pm}, p_3^{\pm} \ldots$$

Proposition 3.1 (*Proposition 1.2 in [17]*). *If the $C(x) \geq c\,|x-\overline{x}|^2\,\mathbf{1}$, then the eigenvalues of H_s^+ concentrate near the eigenvalues of the model operator K^+. That is, for any positive integer q there exists $s_0 > 0$ and $c_1 > 0$ such that for any $s > s_0$*

Theorem 3.2. (1) *there are precisely p_j^+ eigenvalues (multiplicities counted) of H_s^+ in the interval*

$$\left(\mu_j^+ - c_1 s^{-1/5}, \mu_j^+ + c_1 s^{-1/5}\right), \quad j = 1, \ldots, q;$$

(2) *there no eigenvalues of H_s^+ in $\left(-\infty, \mu_1^+ - c_1 s^{-1/5}\right)$ and in the intervals*

$$\left(\mu_j^+ + c_1 s^{-1/5}, \mu_{j+1}^+ - c_1 s^{-1/5}\right), \quad j = 1, \ldots, q;$$

(3) *similar results also hold for operators H_s^- and K^-.*

Corollary 3.3. *In the notation above,*

$$\operatorname{ind}(D) = \dim\ker\left(K^+\right) - \dim\ker\left(K^-\right).$$

Proof. For each $s > 0$, operators $H_s^+ = s^{-1} D_s^- D_s^+$ and $H_s^- = s^{-1} D_s^+ D_s^-$ are positive elliptic self-adjoint operators acting on sections of vector bundles over a compact smooth manifold M. Therefore the operators H_s^+ and H_s^- have discrete spectra $\sigma\left(H_s^{\pm}\right) \subset [0, +\infty)$ with finite multiplicities. By Proposition 3.1, the spectra of K^+ and K^- are also nonnegative (and of course discrete).

Choose any real number $r > 0$, so that r is strictly less than the least positive number in the union of the spectra of K^+ and K^-. Then for any $s > 0$ we have

$$\begin{aligned}
\operatorname{ind}(D) &= \dim\ker\left(s^{-1}\,(D_s)^2\Big|_{\Gamma(M,E^+)}\right) - \dim\ker\left(s^{-1}\,(D_s)^2\Big|_{\Gamma(M,E^-)}\right),\\
&= \dim\ker H_s^+ - \dim\ker H_s^-\\
&= \#\left\{\sigma\left(H_s^+\right) \cap [0,r)\right\} - \#\left\{\sigma\left(H_s^-\right) \cap [0,r)\right\},
\end{aligned}$$

because D_s^+ is an isomorphism between the eigenspaces of H_s^+ and of H_s^- corresponding to nonzero eigenvalues. By choosing s sufficiently large in the formula above and applying Proposition 3.1, we obtain

$$\operatorname{ind}(D) = \dim\ker\left(K^+\right) - \dim\ker\left(K^-\right). \quad \square$$

Remark 3.4. With the notation of Section 2.1, if $Z := \left(Z^+, \left(Z^+\right)^*\right) \in \Gamma\left(M, \operatorname{End}\left(E^+ \oplus E^-\right)\right)$ is everywhere nonsingular and anticommutes with Clifford multiplication by vectors, then the corollary implies that the index of the Dirac operator D must be zero. Proposition 2.20 then yields a new proof that the index of a Dirac operator (and thus any elliptic differential operator) on an odd-dimensional manifold is zero.

By applying Corollary 3.3, Proposition 2.7, and the above remark to even-dimensional manifolds, we obtain the following corollary.

Corollary 3.5. *Let M be even-dimensional and* spin^c. *Let $E^{\pm} \cong (\mathbb{S} \otimes W)^{\pm} = \left(\mathbb{S}^+ \otimes W^{\pm}\right) \oplus \left(\mathbb{S}^- \otimes W^{\mp}\right)$ be any graded, self-adjoint Clifford module over M. Then the index of the Dirac operator corresponding to this Clifford module is zero if the bundles W^+ and W^- are isomorphic.*

Proof. If there exists a bundle isomorphism $\phi^+ : W^+ \to W^-$, then $Z^{\pm} = \gamma \otimes \left(\phi^+, \left(\phi^+\right)^*\right) : E^{\pm} \to E^{\mp}$ is a bundle isomorphism that anticommutes with Clifford multiplication. Therefore, the index is zero. $\square$

4. Local index theory on $\mathbb{R}^n$

In this section, if P is a linear operator defined on a dense domain in a Hilbert space, then P^* denotes the formal adjoint of P.

Let $A_1, \ldots, A_n, B_1, \ldots, B_n$ be $m \times m$ matrices, and let

$$A = \sum_{j=1}^{n} A_j \partial_j, \qquad B(x) = \sum_{j=1}^{n} x_j B_j, \quad \text{and}$$
$$Q = A + B(x).$$

We take the domain of Q to be the space of compactly supported, smooth $\mathbb{C}^m$-valued functions on $\mathbb{R}^n$. We assume that

(1) A is an elliptic operator.
(2) There is a positive constant K such that $(B(x))^* B(x) \geq K|x|^2$ for all $x \in \mathbb{R}^n$.
(3) For each j and k, $A_j^* B_k - B_k^* A_j = 0$.

Remark 4.1. Note that the second condition above is equivalent to the fact that the smallest eigenvalue of $(B(x))^* B(x)$ is at least $K|x|^2$. Thus, the same inequality holds for $B(x)(B(x))^*$.

Then

$$Q^*Q = A^*A - \left(\sum_{j=1}^{n} A_j^* B_j\right) + (B(x))^* B(x), \quad \text{and}$$

$$QQ^* = AA^* + \left(\sum_{j=1}^{n} A_j B_j^*\right) + B(x)(B(x))^*.$$

By [6, Theorem 2.2], the operator $\mathbf{Q}$ on $\mathbb{C}^n \times \mathbb{C}^n$ valued-functions defined by

$$\mathbf{Q} = \begin{pmatrix} 0 & Q^* \\ Q & 0 \end{pmatrix} = \sum_{j=1}^{n} \begin{pmatrix} 0 & -A_j^* \\ A_j & 0 \end{pmatrix} \partial_j + \begin{pmatrix} 0 & (B(x))^* \\ B(x) & 0 \end{pmatrix}$$

is essentially self-adjoint. Its unique self-adjoint extension is defined as the closure of $\mathbf{Q}$ with respect to the inner product norm $\|\cdot\|_{\mathbf{Q}}$ defined by $\|u\|_{\mathbf{Q}}^2 = \|u\|_{L^2}^2 + \|\mathbf{Q}u\|_{L^2}^2$.

Observe that

$$\mathbf{Q}^2 = \begin{pmatrix} Q^*Q & 0 \\ 0 & QQ^* \end{pmatrix}.$$

Each of the operators Q^*Q and QQ^* are bounded from below by

$$P = c\sum_{j=1}^{n} \left(-\partial_j^2 + x_j^2\right)\mathbf{1} - \lambda\mathbf{1}$$

and above by

$$P' = \frac{1}{c}\sum_{j=1}^{n} \left(-\partial_j^2 + x_j^2\right)\mathbf{1} + \lambda\mathbf{1}$$

for some constant $c > 0$, and where λ is the largest eigenvalue of $\left|\sum_{j=1}^{n} A_j B_j^*\right|$. Therefore, the norm $\|\cdot\|_{\mathbf{Q}}$ defined above is equivalent to the harmonic oscillator norm $\|\cdot\|$ defined by

$$\begin{aligned} \|u\|^2 &= \|u\|_{L^2}^2 + \sum_{j=1}^{n} \left\|\partial_j u\right\|_{L^2}^2 + \sum_{j=1}^{n} \left\|x_j u\right\|_{L^2}^2 \\ &= \|u\|_{L^2}^2 + \sum_{j=1}^{n} \left\langle \left(-\partial_j^2 + x_j^2\right) u(x), u(x)\right\rangle_{L^2}. \end{aligned}$$

Denote by $\mathcal{H}$ the closure of the space of compactly supported, smooth $\mathbb{C}^m$-valued functions on $\mathbb{R}^n$ with respect to the norm $\|u\|$.

It is well known that $P + \tau \mathbf{1}$ has a compact inverse for some constant τ, so $\mathbf{Q}^2 + \tau \mathbf{1}$ must also have a compact inverse. It follows that $\mathbf{Q}^2$ has a finite dimensional kernel. We conclude that both operators $\mathbf{Q}$ and Q are Fredholm.

We summarize the arguments of this section in the following proposition.

Proposition 4.2. *Suppose that $Q = \sum_{j=1}^{n} A_j \partial_j + B(x)$ satisfies conditions (1) through (3) at the beginning of this section. Then the closure of the elliptic operator Q is Fredholm on its domain $\mathcal{H}$.*

Corollary 4.3. *Let $\{Q_t | t \in [0,1]\}$ be a family of operators of the form*

$$Q_t = \sum_{j=1}^{n} A_j(t)\,\partial_j + \sum_{j=1}^{n} x_j B_j(t),$$

where $A_j(t)$ and $B_j(t)$ are continuous families of matrices such that $\sum_{j=1}^{n} x_j A_j$ and $\sum_{j=1}^{n} x_j B_j$ are invertible for any nonzero $x \in \mathbb{R}^n$, satisfying

$$A_j^* B_k - B_k^* A_j = 0$$

for each j and k. Then the index of Q_t is defined and is independent of t.

Proof. It suffices to show that the family $\left\{Q_t : \mathcal{H} \to L^2 | t \in [0,1]\right\}$ is a continuous family in the norm topology. If $\max_j \left|A_j(t_1) - A_j(t_2)\right|$ and $\max_j \left|B_j(t_1) - B_j(t_2)\right|$ are both less than $\delta > 0$ (with respect to a fixed matrix norm), then for every $u \in \mathcal{H}$,

$$\begin{aligned}\left\|\left(Q_{t_1} - Q_{t_2}\right)u\right\|_{L^2}^2 &= \left\|\left(\sum_{j=1}^{n}\left(A_j(t_1) - A_j(t_2)\right)\partial_j + \sum_{j=1}^{n} x_j\left(B_j(t_1) - B_j(t_2)\right)\right)u\right\|_{L^2}^2 \\ &\le \delta^2\left(\sum_{j=1}^{n}\left\|\partial_j u\right\|_{L^2}^2 + \sum_{j=1}^{n}\left\|x_j u\right\|_{L^2}^2\right) \le \delta^2\|u\|^2. \quad \square\end{aligned}$$

5. Local calculations

We now proceed with a calculation near the singular sets that evaluates the index of the Dirac operator D. Suppose that Z is a proper perturbation of D on $\Gamma(M, E)$ with a nonempty set of singular points. This implies that there are restrictions on the rank of E on the associated grading and on the graded bundle map Z; see Section 2.4 and Appendix B.8. Let U be the neighborhood of a singular point $\overline{x}$ as in Definition 2.14. We consider the operator $D_s = D + sZ$ on sections of E, which has the local form

$$D_s = D + s\sum_j (x - \overline{x})_j Z_j,$$

where each $Z_j^+ : E^+ \to E^-$ is a locally defined, self-adjoint bundle isomorphism with $Z_j = \left(Z_j^+, \left(Z_j^+\right)^*\right)$.

We choose geodesic normal coordinates x around $\overline{x}$ such that the metric is the identity matrix at $\overline{x}$ and such that ∂_j commutes with $c(\partial_k)$ for all j and k at the point $x = \overline{x}$. We wish to calculate the dimensions of $\ker\left(D_s^2|_{\Gamma(M,E^\pm)}\right)$. By Corollary 3.3, it suffices to calculate the dimensions of the kernels of the model operators $K^\pm(\overline{x})$ corresponding to each singular point $\overline{x}$. In the notation of Section 3, the model operator is $K(\overline{x}) = -A^{(2)} + \overline{B} + C^{(2)}$, with

$$A^{(2)} = \sum_{j=1}^{n} \mathbf{1}\partial_j^2,$$

$$\overline{B} = Z \circ D + D \circ Z|_{\overline{x}} = \sum_{j=1}^{n} c\left(\partial_j\right) Z_j(\overline{x}), \qquad \text{and}$$

$$C^{(2)} = \text{quadratic part of } Z^2 = \left(\sum_j (x - \overline{x})_j Z_j(\overline{x})\right)^2.$$

Notice that

$$K(\overline{x}) = \begin{pmatrix} K^{+}(\overline{x}) & 0 \\ 0 & K^{-}(\overline{x}) \end{pmatrix} = \begin{pmatrix} D^{-}(\overline{x})\, D^{+}(\overline{x}) & 0 \\ 0 & D^{+}(\overline{x})\, D^{-}(\overline{x}) \end{pmatrix},$$

where the operator $D(\overline{x})$ is defined by

$$D(\overline{x}) = \sum_j c\left(\partial_j\right)\partial_j + \sum_j (x-\overline{x})_j\, Z_j(\overline{x}),$$

which satisfies the hypothesis of Proposition 4.2. Thus, $D(\overline{x})$ is a Fredholm operator on $\mathbb{R}^n$, and

$$\dim\ker\left(K^{+}(\overline{x})\right) - \dim\ker\left(K^{-}(\overline{x})\right) = \operatorname{ind}_{\mathbb{R}^n}(D(\overline{x})).$$

Corollary 3.3 implies the following theorem.

Theorem 5.1. *Let $D : \Gamma(M,E) \to \Gamma(M,E)$ be the Dirac operator corresponding to a graded, self-adjoint, Hermitian Clifford module over a closed manifold. Suppose that there exists a proper perturbation (Definition 2.14) Z of D. Then near each singularity $\overline{x}$ of Z, we write $Z(x) = \sum_j (x-\overline{x})_j Z_j$, where each $Z_j^{+} : E^{+} \to E^{-}$ is a locally defined bundle isomorphism with $Z_j = \left(Z_j^{+}, \left(Z_j^{+}\right)^{*}\right)$. Then the index of D satisfies*

$$\operatorname{ind}(D) = \sum_{\overline{x}\ \text{singular}} \operatorname{ind}_{\mathbb{R}^n}(D(\overline{x})),$$

where $D(\overline{x}) = \sum_j c\left(\partial_j\right)\partial_j + \sum_j x_j Z_j(\overline{x})$ is the operator on $\mathbb{R}^n$ that maps $E^{+}_{\overline{x}}$-valued functions to $E^{-}_{\overline{x}}$-valued functions.

Let us now specialize to a case where we compute the local indices $\operatorname{ind}_{\mathbb{R}^n}(D(\overline{x}))$ in terms of the matrices $c(\partial_1), \ldots, c(\partial_n), Z_1(\overline{x}), \ldots, Z_n(\overline{x})$ directly. We will need to assume that $Z(x)^2 = q(x-\overline{x})\,\mathbf{1}$, where q is a positive definite quadratic form and $\mathbf{1}$ is the identity map. The following propositions show that there always exist local bundle maps with this property. Furthermore, the coordinates may be chosen so that the $Z_j(\overline{x})$ anticommute.

Proposition 5.2. *Suppose the operator $Z(x) = \sum_{j=1}^{n}(x-\overline{x})_j Z_j$ satisfies $Z(x)^2 = q(x-\overline{x})\,\mathbf{1}$, where q is a positive definite quadratic form and $\mathbf{1}$ is the identity map. Then there exist local coordinates y and Hermitian linear transformations $\widetilde{Z}_2, \ldots, \widetilde{Z}_n$ such that*

$$Z(y) = \sum_{j=1}^{n} y_j \widetilde{Z}_j$$

and $\widetilde{Z}_j\widetilde{Z}_k + \widetilde{Z}_k\widetilde{Z}_j = 2\delta_{jk}\mathbf{1}$. Furthermore, we have that $E \cong \mathbb{S}\otimes W$, where the rank of W is a multiple of $2^{\left\lceil\frac{n-1}{2}\right\rceil}$.

Proof. If $Z(x)^2 = q(x-\overline{x})\,\mathbf{1}$, then there is a symmetric matrix Q, an orthogonal matrix U, and a positive diagonal matrix D such that

$$q(x-\overline{x}) = Q(x-\overline{x})\cdot(x-\overline{x})$$

and such that $D = UQU^{\mathrm{T}}$ is the identity. Let $y = \sqrt{D}U(x-\overline{x})$. In the new coordinates we have

$$Z(y) = \sum_{j=1}^{n} y_j \widetilde{Z}_j$$

for the Hermitian linear transformations $\widetilde{Z}_j = \sum_{k=1}^{n}\left(\sqrt{D}^{-1}U\right)_{jk} Z_k$. Then $\left(Z^2\right)(y) = \left(\|y\|^2\right)I$ implies the following relations:

$$\widetilde{Z}_j\widetilde{Z}_k + \widetilde{Z}_k\widetilde{Z}_j = 2\delta_{jk}.$$

Note then that $\left\{\mathrm{i}\widetilde{Z}_j\right\}$ becomes a set of Clifford matrices, all of which commute with the given Clifford action. □

Proposition 5.3. *Suppose that $r = \operatorname{rank}\left(E^{+}\right)$ is a multiple of 2^{n-1}. Then there exists a set of Hermitian, invertible linear transformations*

$$\left\{ Z_j = \begin{pmatrix} 0 & Z_j^- \\ Z_j^+ & 0 \end{pmatrix} := \begin{pmatrix} 0 & \left(Z_j^+\right)^* \\ Z_j^+ & 0 \end{pmatrix} \right\}_{1\le j\le n}$$

on the graded $\mathbb{C}\mathrm{l}\left(\mathbb{R}^n\right)$ module $E^+ \oplus E^- \cong \mathbb{C}^r \oplus \mathbb{C}^r$ such that

(1) *each Z_j anticommutes with each $c\left(\partial_k\right)$ for $1 \le k \le n$, and*
(2) *The operator $Z(x) = \sum_{j=1}^{n} (x-\overline{x})_j Z_j$ satisfies $Z(x)^2 = q(x-\overline{x})\,\mathbf{1}$, where q is a positive definite quadratic form and $\mathbf{1}$ is the identity map.*

Proof. Suppose that the rank of E^+ is a multiple of 2^{n-1}, and E is endowed with a graded $\mathbb{C}\mathrm{l}\left(\mathbb{R}^n\right)$ action. Then there exists a graded $\mathbb{C}\mathrm{l}\left(\mathbb{R}^{2n}\right)$ action extending the original action on $E = E^+ \oplus E^-$. If $\left\{\beta_j\right\}_{1\le j\le n}$ is a set of generators corresponding to Clifford multiplication on E by the additional vectors, then $\left\{Z_j = \mathrm{i}\beta_j\right\}_{1\le j\le n}$ is a set of transformations that satisfy the conditions of the proposition. □

If the $\left\{Z_j\left(\overline{x}\right)\right\}$ anticommute (changing coordinates if necessary), then

$$\begin{aligned} K\left(\overline{x}\right) &= -\sum_{j=1}^{n} \partial_j^2 + \sum_{j=1}^{n} c\left(\partial_j\right) Z_j + \left(\sum_j (x-\overline{x})_j Z_j\right)^2 \\ &= -\sum_{j=1}^{n} \partial_j^2 + \sum_{j=1}^{n} c\left(\partial_j\right) Z_j + \sum_{j=1}^{n} (x-\overline{x})_j^2 Z_j^2 \\ &= \sum_{j=1}^{n} \left(-\partial_j^2 + c\left(\partial_j\right) Z_j + (x-\overline{x})_j^2 \left(c\left(\partial_j\right) Z_j\right)^2\right), \end{aligned}$$

where each Z_j is evaluated at $x = \overline{x}$.

Observe that the operators $L_j = c\left(\partial_j\right) Z_j$ are Hermitian and commute with each other and thus can be diagonalized simultaneously. Let v be a common eigenvector of each operator L_j corresponding to the eigenvalue λ_j. Letting f be a function of x, we have

$$K\left(\overline{x}\right)(fv) = \left(\sum_{j=1}^{n} \left(-\partial_j^2 + \lambda_j + \lambda_j^2 (x-\overline{x})_j^2\right) f\right) v.$$

The section fv is in the kernel of $K\left(\overline{x}\right)$ if and only if each λ_j is negative, and, up to a constant, $fv = \exp\left(\frac{1}{2}\sum_j \lambda_j (x-\overline{x})_j^2\right) v$. Thus the dimension of the kernel of $K\left(\overline{x}\right)$ is the dimension of the intersection of the direct sum of eigenspaces $E_\lambda\left(L_j\right)$ of $L_j = c\left(\partial_j\right) Z_j$ corresponding to negative eigenvalues. Note that L_j maps E^+ to itself (call the restriction L_j^+), so that the dimension of the kernel of D_s^+ restricted to this neighborhood is simply the dimension of $\bigcap_j \left(\bigoplus_{\lambda<0} E_\lambda \left(L_j^+\right)\right)$.

The calculation above and Corollary 3.3 imply the following theorem.

Theorem 5.4. *Let $D : \Gamma(M, E) \to \Gamma(M, E)$ be the Dirac operator corresponding to a graded, self-adjoint, Hermitian Clifford module over a closed manifold. Suppose that there exists a proper perturbation (Definition 2.14) Z of D. Then near each singularity $\overline{x}$ of Z, we write $Z(x) = \sum_j (x-\overline{x})_j Z_j$, where each $Z_j^+ : E^+ \to E^-$ is a locally defined bundle isomorphism with $Z_j = \left(Z_j^+, \left(Z_j^+\right)^*\right)$. We assume that we may choose Z such that $Z_j\left(\overline{x}\right) Z_k\left(\overline{x}\right) = -Z_k\left(\overline{x}\right) Z_j\left(\overline{x}\right)$ for all $j \ne k$. Define the Hermitian linear transformations*

$$L_j^{\pm}\left(\overline{x}\right) = c\left(\partial_j\right) Z_j|_{E_{\overline{x}}^{\pm}}.$$

Then

$$\operatorname{ind}(D)=\sum_{\overline{x}}\left(\dim\left[\bigcap_j\left(\bigoplus_{\lambda<0}E_\lambda\left(L_j^+(\overline{x})\right)\right)\right]-\dim\left[\bigcap_j\left(\bigoplus_{\lambda<0}E_\lambda\left(L_j^-(\overline{x})\right)\right)\right]\right),$$

where the sum is taken over all the singular points $\overline{x}$ of Z.

Remark 5.5. The assumption that $Z_j(\overline{x})Z_k(\overline{x})=-Z_k(\overline{x})Z_j(\overline{x})$ for all $j\neq k$ is natural, since it appears in all local calculations where researchers have used Witten deformation (see papers mentioned in the introduction). In fact, a stronger condition, that Z_j^2 is a scalar multiplication by a function and that the anticommutivity holds for all x near $\overline{x}$, appears in all previous work.

We now apply Theorem 5.4 when $n=\dim M$ is even. Suppose that the hypotheses of Theorem 2.17 hold, and suppose that Z has a nonempty set of singular points. Let U be the neighborhood of a singular point $\overline{x}$ as in Definition 2.14. Then the operator $D_s=D+sZ$ over $E|_U\cong\mathbb{S}|_U\otimes W$ has the form

$$\begin{aligned}D_s&=D+s\gamma\otimes\phi\\&=D+s\gamma\otimes\sum_j(x-\overline{x})_j\,\phi_j,\end{aligned}$$

with notation as in Definition 2.14 and Proposition 2.7, where each $\phi_j^+:W^+\to W^-$ is a locally defined self-adjoint bundle isomorphism with $\phi_j=\left(\phi_j^+,\left(\phi_j^+\right)^*\right)$. The theorem becomes:

Corollary 5.6. *Let $D:\Gamma(M,E)\to\Gamma(M,E)$ be the Dirac operator corresponding to a graded, self-adjoint, Hermitian Clifford module over an even-dimensional manifold. Let Z be a proper perturbation of D. Then near each singularity $\overline{x}$ of Z, we write $E^\pm\cong\left(S^\pm\otimes W^\pm\right)\oplus\left(S^\pm\otimes W^\pm\right)$ and $Z(x)=\sum_j(x-\overline{x})_j\,\gamma\otimes\phi_j$, where each $\phi_j^+:W^+\to W^-$ is a locally defined bundle isomorphism with $\phi_j=\left(\phi_j^+,\left(\phi_j^+\right)^*\right)$. Assume that $\phi_j\phi_k=-\phi_k\phi_j$ for all $j\neq k$. Define the Hermitian linear transformations*

$$L_j^\pm(\overline{x})=\left(\left(c\left(\partial_j\right)\gamma\right)\otimes\phi_j\right)|_{E_{\overline{x}}^\pm}.$$

Then

$$\operatorname{ind}(D)=\sum_{\overline{x}}\left(\dim\left[\bigcap_j\left(\bigoplus_{\lambda<0}E_\lambda\left(L_j^+(\overline{x})\right)\right)\right]-\dim\left[\bigcap_j\left(\bigoplus_{\lambda<0}E_\lambda\left(L_j^-(\overline{x})\right)\right)\right]\right),\tag{5.1}$$

where the sum is taken over all the singular points $\overline{x}$ of Z.

Remark 5.7. The proof is easily modified for the odd-dimensional case. Locally we write $E|_U\cong S\otimes\left(W'\oplus W'\right)|_U$ as in Proposition 2.10. Then $Z=\sum_j(x-\overline{x})_j\,\mathbf{1}\otimes\begin{pmatrix}0&\phi_j\\-\phi_j&0\end{pmatrix}$, where each $\phi_j:W'\to W'$ is a locally defined, skew-adjoint bundle isomorphism. Formula (5.1) is valid with the new $L_j^\pm(\overline{x})$ given by

$$L_j^\pm(\overline{x})=c^+\left(\partial_j\right)\otimes\left.\begin{pmatrix}0&\phi_j\\\phi_j&0\end{pmatrix}\right|_{E_{\overline{x}}^\pm}.$$

In this case, the formula shows that the sum of the local indices is always zero, since $\operatorname{ind}(D)=0$ in odd dimensions.

6. Examples

6.1. The de Rham operator

The bundle $E=\Lambda^\bullet T^*M\otimes\mathbb{C}$ of complex-valued forms is a left Clifford module with the canonical Clifford action defined by

$$l(v)\omega=v^\flat\wedge\omega-v\lrcorner\omega,\quad v\in T_xM,\ \omega\in\Lambda^\bullet T_x^*M\otimes\mathbb{C}.$$

Here $v^\flat$ denotes the covector dual to v and $v\lrcorner$ is the contraction with vector v. Similarly, E is also a right Clifford module with the canonical right Clifford action on $\Lambda^p T^*M \otimes \mathbb{C}$ defined by

$$r(v)\omega = (-1)^p \left(v^\flat \wedge \omega + v\lrcorner\omega\right).$$

The corresponding Dirac operator is the *de Rham operator*

$$D = \sum l\left(e_j\right) \nabla_{e_j} = d + d^*,$$

where $\{e_1, \dots, e_n\}$ is a local orthonormal basis of TM.

Let the dimension of M be even. Let $\mathbb{S}$ denote a spin^c bundle, which always exists locally but may not exist globally if M is not a spin^c manifold. Let $\mathbb{S}^*$ denote the dual bundle to $\mathbb{S}$. Using a bundle metric on $\mathbb{S}$, identify $\mathbb{S}^*$ and $\overline{\mathbb{S}}$. The result below follows from the representation theory of Clifford algebras.

Lemma 6.1 (*Follows from Ch. IV in [9]*). *Through the isomorphism of Clifford modules $\mathbb{S} \otimes \mathbb{S}^* \cong \Lambda^\bullet T^*M \otimes \mathbb{C}$, the corresponding Clifford actions by vectors are intertwined in the following commutative diagrams:*

$$\begin{array}{cccc|cccc}
\mathbb{S}\otimes\mathbb{S}^* & \xrightarrow{\cong} & \Lambda^\bullet T^*M\otimes\mathbb{C} & & \mathbb{S}\otimes\mathbb{S}^* & \xrightarrow{\cong} & \Lambda^\bullet T^*M\otimes\mathbb{C} \\
\downarrow c(v)\otimes 1 & \circlearrowleft & \downarrow l(v) & & \downarrow 1\otimes c(-v) & \circlearrowleft & \downarrow r(v) \\
\mathbb{S}\otimes\mathbb{S}^* & \xrightarrow{\cong} & \Lambda^\bullet T^*M\otimes\mathbb{C} & & \mathbb{S}\otimes\mathbb{S}^* & \xrightarrow{\cong} & \Lambda^\bullet T^*M\otimes\mathbb{C}.
\end{array}$$

The grading of E into even and odd forms $E = E^+ \oplus E^-$, where $E^+ = \Lambda^{\mathrm{even}} T^*M \otimes \mathbb{C}$ and $E^- = \Lambda^{\mathrm{odd}} T^*M \otimes \mathbb{C}$, is not natural; it does not come from the action of the chirality operator γ on E. Under the isomorphism $E = \Lambda^\bullet T^*M \otimes \mathbb{C} \cong \mathbb{S} \otimes \mathbb{S}^*$ we have $E^+ = \left(\mathbb{S}^+ \otimes \left(\mathbb{S}^+\right)^*\right) \oplus \left(\mathbb{S}^- \otimes \left(\mathbb{S}^-\right)^*\right)$ and $E^- = \left(\mathbb{S}^+ \otimes \left(\mathbb{S}^-\right)^*\right) \oplus \left(\mathbb{S}^- \otimes \left(\mathbb{S}^+\right)^*\right)$. Since the Dirac operator only acts on the first component of $\mathbb{S} \otimes \mathbb{S}^*$, we can deform D by any linear operator $Z = \gamma \otimes \phi$, where γ is the chirality operator and $\phi : \mathbb{S}^\pm \to \mathbb{S}^\mp$ is a bundle map as in Proposition 2.7.

The following example of such a deformation is useful in the proof of the Poincaré–Hopf Theorem. Let V be a smooth vector field on M, then for each p-form ω define

$$Z_V\omega = \left(V^\flat \wedge + V\lrcorner\right)\omega = (-1)^p\, r(V)\,\omega : E^\pm \to E^\mp.$$

Recall that a point $\overline{x} \in M$ is called a *singular (or critical)* point of $V(x) = \sum_k V_k(x)\,\partial_k$ if for all $k = 1, \dots, n$, $V_k(\overline{x}) = 0$. A singular point of V is called *nondegenerate* if $\det\left(\partial V_k/\partial x_i\right)(\overline{x}) \neq 0$. The index of a singular point $\overline{x}$ is defined as follows

$$\mathrm{ind}_V(\overline{x}) = \operatorname{sign}\det\left(\partial V_k/\partial x_i\right)(\overline{x}).$$

Remark 6.2. Note that $\overline{x} \in M$ is a nondegenerate singular point of V if and only if $\overline{x}$ is a proper singular point of the endomorphism Z_V on forms.

Observe that the map $\omega \longmapsto (-1)^p\,\omega$ on p-forms ω is given by the map $\gamma \otimes \gamma$ on $\mathbb{S} \otimes \mathbb{S}^* = \mathbb{S} \otimes \overline{\mathbb{S}}$, since γ is the identity (respectively, minus the identity) on even (respectively, odd) spinors. Thus,

$$Z_V = (-1)^p\, r(V) = (\mathbf{1} \otimes c(V)) \circ (\gamma \otimes \gamma) = (\gamma \otimes c(V)\,\gamma)$$

so that $Z_V = \gamma \otimes \phi$ with $\phi = c(V)\,\gamma$.

Near each nondegenerate singular point $\overline{x}$ of the vector field V, we choose local coordinates centered at $\overline{x}$ so that $V(x) = \sum_{j=1}^n x_j \sum_{k=1}^n V_{jk}\partial_k + \mathcal{O}\left(|x|^2\right)$, where $\left(V_{jk}\right)$ is an invertible matrix. The model operator $K(\overline{x})$ from Section 5 is

$$K(\overline{x}) = -\sum_{j=1}^n \mathbf{1}\partial_j^2 + \sum_{j,k=1}^n V_{jk}\left(c\left(\partial_j\right)\gamma \otimes c\left(\partial_k\right)\gamma\right) + \sum_{j,k,m=1}^n x_j x_k \left(VV^{\mathrm{T}}\right)_{jk}.$$

We observe that $K(\overline{x}) = D(\overline{x})^2$ on even forms, where

$$D(\overline{x}) : \Gamma\left(\mathbb{R}^n, \Lambda^{\mathrm{even/odd}} T^*\mathbb{R}^n \otimes \mathbb{C}\right) \to \Gamma\left(\mathbb{R}^n, \Lambda^{\mathrm{odd/even}} T^*\mathbb{R}^n \otimes \mathbb{C}\right)$$

is defined by

$$D(\overline{x}) = \sum_{j=1}^{n} \left(c\left(\partial_j\right) \otimes \mathbf{1}\right) \partial_j + \sum_{j,k=1}^{n} x_j V_{jk}(\gamma \otimes c(\partial_k)\gamma).$$

By Theorem 5.1,

$$\chi(M) = \operatorname{ind}\left(d + d^*\right) = \sum_{V(\overline{x})=0} \operatorname{ind}_{\mathbb{R}^n}(D(\overline{x})). \tag{6.1}$$

By Corollary 4.3, each of the integers $\operatorname{ind}_{\mathbb{R}^n}(D(\overline{x}))$ does not change if the vector field V is deformed continuously while remaining nondegenerate and without introducing additional zeros. Every vector field on $\mathbb{R}^n$ with a single, nondegenerate zero at the origin may be continuously deformed in this way to $V(x) = \pm x_1 \partial_1 + \sum_{j=2}^{n} x_j \partial_j$, depending only on the index ± 1. Thus, to evaluate the right hand side of Eq. (6.1), it suffices to calculate the index of the corresponding operator

$$D(\overline{x}) = \sum_{j=1}^{n} \left(c\left(\partial_j\right) \otimes \mathbf{1}\right) \partial_j \pm x_1 \left(\gamma \otimes c\left(\partial_1\right) \gamma\right) + \sum_{j=2}^{n} x_j \left(\gamma \otimes c\left(\partial_j\right) \gamma\right).$$

The operator above satisfies the conditions of Theorem 5.4. Note that the local zeroth order operator Z corresponds to

$$\begin{aligned} Z_V &= \gamma \otimes \phi = \gamma \otimes \left[\pm x_1 c\left(\partial_1\right) \gamma + \sum_{j=2}^{n} x_j \left(c(\partial_j)\gamma\right)\right] \\ &= \gamma \otimes \left[\sum_{j=1}^{n} x_j \phi_j\right]. \end{aligned}$$

In the notation of Section 5 we have

$$\begin{aligned} L_1 &:= \pm \left(c\left(\partial_1\right) \gamma\right) \otimes \left(c\left(\partial_1\right) \gamma\right) \\ L_j &:= \left(c\left(\partial_j\right) \gamma\right) \otimes \left(c\left(\partial_j\right) \gamma\right) \quad \text{for } j \geq 2. \end{aligned}$$

If ω is a p-form, then for instance

$$\begin{aligned} L_j \omega &= (-1)^p l\left(\partial_j\right) r\left(\partial_j\right) \omega \\ &= \left(\mathrm{d}x_j \wedge - \mathrm{d}x_j \lrcorner\right) \left(\mathrm{d}x_j \wedge + \mathrm{d}x_j \lrcorner\right) \omega \quad \text{for } j \geq 2. \end{aligned}$$

If, in addition, $\omega = \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_p}$, then

$$L_j \omega = \begin{cases} \omega, & \text{if } j \in \left\{i_1, \ldots, i_p\right\} \text{ and } j \geq 2 \\ -\omega, & \text{if } j \notin \left\{i_1, \ldots, i_p\right\} \text{ and } j \geq 2 \\ \pm\omega, & \text{if } j = 1 \in \left\{i_1, \ldots, i_p\right\} \\ \mp\omega, & \text{if } j = 1 \notin \left\{i_1, \ldots, i_p\right\}. \end{cases}$$

The proposition below follows.

Proposition 6.3. *In a neighborhood of a zero of index ± 1, then the form ω belongs to the negative eigenspace of L_j for each j if and only if*

$$\omega = \begin{cases} c & \text{if } \operatorname{ind}_V(\overline{x}) = 1 \\ c\,\mathrm{d}x_1 & \text{if } \operatorname{ind}_V(\overline{x}) = -1 \end{cases}$$

for some constant c.

Since $(\Lambda^{\bullet} T^*\mathbb{R}^n \otimes \mathbb{C})^+ = (\Lambda^{\text{even}} T^*\mathbb{R}^n \otimes \mathbb{C})$, the local indices of $D^{\pm}(\overline{x})$ — and thus the index $\operatorname{ind}_{\mathbb{R}^n}(D(\overline{x}))$ — are now clearly determined.

Corollary 6.4. *We have* $\mathrm{ind}_{\mathbb{R}^n}(D(\overline{x})) = \mathrm{ind}_V(\overline{x})$.

The Poincaré–Hopf Theorem now follows from Eq. (6.1).

Theorem 6.5 (*Poincaré–Hopf Theorem*). *Let V be a smooth vector field with nondegenerate singular points on an even-dimensional, smooth manifold M. Let $n^{\pm}$ denote the number of singular points of V with the index ± 1. Then*

$$\chi(M) = n^{+} - n^{-},$$

where $\chi(M)$ is the Euler characteristic of M.

Remark 6.6. The Poincaré–Hopf Theorem on odd-dimensional manifolds can be proved in a similar way.

Corollary 6.7. *The Euler characteristic of a closed, even-dimensional* spin^c *manifold is zero if and only if the* spin^c *bundles $\mathbb{S}^{+}$ and $\mathbb{S}^{-}$ are isomorphic. The statement is true for any choice of* spin^c *bundle $\mathbb{S}$.*

Proof. If the Euler characteristic of the manifold is zero, then there exists a nonzero vector field. Clifford multiplication by this vector field provides the needed isomorphism. On the other hand, if a bundle isomorphism $\psi : \mathbb{S}^{+} \to \mathbb{S}^{-}$ does exist, it induces a map $\phi : \mathbb{S}^{*} \to \mathbb{S}^{*}$ defined using $\phi^{+} = \psi^{*} : \left(\mathbb{S}^{+}\right)^{*} \to \left(\mathbb{S}^{-}\right)^{*}$ and $\phi^{-} = \left(\phi^{+}\right)^{*}$ and thus a proper perturbation $Z = \gamma \otimes \phi$ of the de Rham operator with no singular points. □

6.2. The Euler characteristic and sections of the conformal Pin bundle

In this section, we again assume that the dimension n of M is even. We will show that the Euler characteristic is the sum of indices of a nondegenerate section of the conformal Pin bundle.

Consider the subset $P_x M$ of all elements of the $\mathbb{C}\mathrm{l}(T_x M)$ of the form

$$v_1 v_2 \ldots v_r$$

where $v_j \in T_x M$ for each j. This set forms a monoid, and it is the same as

$$P_x M = \{\lambda\alpha \mid \alpha \in \mathrm{Pin}(T_x M), \lambda \in \mathbb{R}\} \subset \mathbb{C}\mathrm{l}(T_x M)$$

(see [9, p. 12ff]). Note that $P_x M - \{0\}$ is the conformal Pin group of $T_x M$. Let PM denote the corresponding bundle of monoids over M. Let $P^{+}M$ and $P^{-}M$ be defined by

$$P^{\pm}M = PM \cap \mathbb{C}\mathrm{l}^{\pm}(T_x M).$$

We say that $\overline{x} \in M$ is a *nondegenerate zero* of $\beta \in \Gamma\left(P^{-}M\right)$ if on a sufficiently small neighborhood U of $\overline{x}$ we have

(1) $\beta|_{\overline{x}} = 0$, and
(2) in local coordinates x on U, there exist invertible $\beta_j \in \Gamma\left(P^{-}M|_U\right)$ for $1 \le j \le n = \dim M$ over U such that $\beta = \sum_j (x - \overline{x})_j \beta_j$ on U, and β is invertible on $U \setminus \{\overline{x}\}$.

If $\overline{x} \in M$ is a nondegenerate zero of $\beta \in \Gamma\left(P^{-}M\right)$, then on some neighborhood U of $\overline{x}$,

$$\beta|_U = W_1 W_2 \ldots W_r|_U$$

for some vector fields $W_1, \ldots, W_r$. Since β is nondegenerate, only one of the vector fields (say W_j) is zero at $\overline{x}$, and $\overline{x}$ is a nondegenerate zero of W_j.

Lemma 6.8. *For any two vector fields B_1 and B_2 such that B_2 is nonzero,*

$$B_2 B_1 = \widetilde{B_1} B_2,$$

where $\widetilde{B_1}$ is the vector field defined by

$$\widetilde{B_1} = B_1^{\parallel} - B_1^{\perp},$$

where $B_1^{\|}$ and $B_1^{\perp}$ are the components of B_1 in directions parallel to and perpendicular to B_2. If B_1 has a nondegenerate zero at a point where B_2 is nonzero, the index of $\widetilde{B_1}$ at the point is the opposite of the index of B_1 at the point.

Proof. The equation follows from the construction of $\widetilde{B_1}$. The vector field $-\widetilde{B_1}$ is the reflection of B_1 in the plane perpendicular to B_2. Since the dimension is even, the index of $\widetilde{B_1}$ is the opposite of the index of B_1. $\square$

Lemma 6.8 implies that $\overline{x} \in M$ is a nondegenerate zero of $\beta \in \Gamma\left(P^{-}M\right)$ if and only if in local coordinates x on a sufficiently small neighborhood U of $\overline{x}$, there exists a locally defined vector field V_1 with an isolated nondegenerate zero at $\overline{x}$ and a collection of nonzero vector fields $V_2, \ldots, V_r$ such that

$$\beta|_U = V_1 V_2 \ldots V_r|_U.$$

Definition 6.9. Given a section $\beta \in \Gamma\left(P^{-}M\right)$ with a nondegenerate zero at $\overline{x} \in M$, we define the *index* $\operatorname{ind}_{\beta}(\overline{x})$ of β at $\overline{x}$ to be the index $\operatorname{ind}_{V_1}(\overline{x})$ of any vector field V_1 at $\overline{x}$ such that in a neighborhood U of $\overline{x}$,

$$\beta|_U = V_1 V_2 \ldots V_r|_U,$$

where V_1 has an isolated nondegenerate zero at $\overline{x}$ and the vector fields $V_2, \ldots, V_r$ are nonzero at $\overline{x}$.

Lemma 6.10. *Given a section $\beta \in \Gamma\left(P^{-}M\right)$ with a nondegenerate zero at $\overline{x} \in M$, the index of β at $\overline{x}$ is well-defined.*

Proof. Suppose that we are given two different local expressions for β on a sufficiently small neighborhood U of $\overline{x}$:

$$\begin{aligned}\beta|_U &= V_1 V_2 \ldots V_r|_U \\ &= W_1 W_2 \ldots W_{r'}|_U,\end{aligned}$$

where V_1 and W_1 have isolated nondegenerate zeros at $\overline{x} = 0$ and the vector fields $V_2, \ldots, V_r, W_2, \ldots, W_{r'}$ are nonzero at 0. Without loss of generality, we replace V_1 and W_1 with their linear parts

$$V_1 = \sum_{j=1}^{n} x_j V_{1j}, \quad W_1 = \sum_{j=1}^{n} x_j W_{1j},$$

and we replace $V_2, \ldots, V_r, W_2, \ldots, W_{r'}$ with their values at 0. Thus, the equation for β above implies that

$$V_{1j} V_2 \ldots V_r = W_{1j} W_2 \ldots W_{r'}.$$

We multiply on the right by the inverse of $V_2 \ldots V_r$ to obtain

$$\begin{aligned}V_{1j} &= W_{1j}\left[\frac{(-1)^{r-1}}{\|V_r\|^2 \ldots \|V_2\|^2} W_2 \ldots W_{r'} V_r \ldots V_2\right] \\ &= W_{1j} T,\end{aligned}$$

where T is defined to be the element $T \in P_{\overline{x}}^{+}M$ shown in the square brackets above. We note that

$$T = \frac{-1}{\|W_{1j}\|^2} W_{1j} V_{1j} \quad \text{for each } j, \tag{6.2}$$

so that it is an element of $P_{\overline{x}}^{+}M$ of degree at most two. Choose an orthonormal basis $\{f_1, \ldots, f_n\}$ of $T_{\overline{x}}M$ such that

$$\begin{aligned}f_1 &= \frac{W_{11}}{\|W_{11}\|}, \\ V_{11} &= c_1 f_1 + c_2 f_2\end{aligned}$$

with $c_1, c_2 \in \mathbb{R}$. From Eq. (6.2) with $j = 1$, it is easily seen that

$$T = t_1 + t_2 f_1 f_2$$

for some constants $t_1, t_2 \in \mathbb{R}$.

Suppose that for some $j > 1$, W_{1j} is not contained in the space spanned by f_1 and f_2, so that

$$W_{1j} = k_1 f_1 + k_2 f_2 + q,$$

where q is a nonzero vector orthogonal to f_1 and f_2. Then

$$\begin{aligned} V_{1j} &= W_{1j} T \\ &= t_1 W_{1j} + t_2 W_{1j} f_1 f_2 \\ &= \left(t_1 W_{1j} - t_2 k_1 f_2 + t_2 k_2 f_1\right) + t_2 q f_1 f_2, \end{aligned}$$

which is a vector if and only if $t_2 = 0$. Thus, if for some $j > 1$ there is a vector W_{1j} orthogonal to the space spanned by V_{11} and W_{11}, then $V_{1k} = t W_{1k}$ for some nonzero real number t, for all k. Since the dimension n is even, this implies that the vector fields V_1 and W_1 have the same index at the origin.

If on the other hand, W_{1j} is contained in the span of f_1 and f_2 for each j, then the dimension n is 2, and the map

$$W_{1j} \longmapsto W_{1j}(t_1 + t_2 f_1 f_2),$$

which transforms each W_{1j} to V_{1j}, induces an orientation-preserving linear transformation, whose matrix in the basis $\{f_1, f_2\}$ is $\begin{pmatrix} t_1 & -t_2 \\ t_2 & t_1 \end{pmatrix}$. Thus, in this case, the indices of the vector fields V_1 and W_1 are the same as well. □

Theorem 6.11. *For any section $\beta \in \Gamma\left(P^- M\right)$ with nondegenerate zeros, the Euler characteristic satisfies*

$$\chi(M) = \sum_{\overline{x}} \mathrm{ind}_\beta(\overline{x})$$

where the sum is taken over all the zeros $\overline{x}$ of β.

Proof. Consider the proper perturbation Z of the de Rham operator

$$D = d + d^* : \Gamma\left(M, \Lambda^{\mathrm{even}} T^* M \otimes \mathbb{C}\right) \to \Gamma\left(M, \Lambda^{\mathrm{odd}} T^* M \otimes \mathbb{C}\right)$$

defined by

$$Z_\beta = \gamma \otimes c(\beta)^* \gamma : \mathbb{S} \otimes \mathbb{S}^* \to \mathbb{S} \otimes \mathbb{S}^*,$$

where we have used the isomorphism $\Lambda^\bullet T^* M \otimes \mathbb{C} \cong \mathbb{S} \otimes \mathbb{S}^*$. In a neighborhood U of a particular zero $\overline{x}$, we write

$$\beta|_U = V_1 V_2 \ldots V_r|_U,$$

where V_1 has an isolated nondegenerate zero at $\overline{x}$ and the vector fields $V_2, \ldots, V_r$ are nonzero at $\overline{x}$. Observe that

$$\begin{aligned} Z_\beta &= r(\beta)(-1)^p \\ &= \left(V_1^\flat \wedge + \mathrm{i}(V_1)\right)\left(V_2^\flat \wedge + \mathrm{i}(V_2)\right) \ldots \left(V_r^\flat \wedge + \mathrm{i}(V_r)\right) \end{aligned}$$

on p-forms (see Section 6.1). We now use Theorem 5.1 to calculate the index of D, which is the Euler characteristic. Without changing the local index $\mathrm{ind}_{\mathbb{R}^n}(D(\overline{x}))$, we will deform it in a particularly simple way near $\overline{x}$. We locally deform the vector fields $V_2, \ldots, V_r$ smoothly to $\frac{V_2}{\|V_2\|}$ (while keeping β invertible on $U \setminus \{\overline{x}\}$), and thus the product $V_2 \ldots V_r$ is deformed to $(-1)^{r(r-1)/2}$. Then β has been deformed to

$$\widetilde{\beta}\big|_U = (-1)^{r(r-1)/2} V_1\Big|_U.$$

The perturbation proof of the Poincaré–Hopf theorem implies that the difference in dimensions of the kernels of the operators $(D + sZ)^\pm\big|_U$ for large s is the index of the vector field $\pm V_1$ at $\overline{x}$, which is the same as the index of the V_1 at $\overline{x}$, which is by definition $\mathrm{ind}_\beta(\overline{x})$. □

Remark 6.12. The theorem above implies that the odd spinor bundle $\mathrm{Pin}^-(TM)$ has a section if and only if the Euler characteristic of M is zero.

Example 6.13. Consider the standard Dirac operator on the (trivial) spin^c bundle $\mathbb{S} = \mathbb{C}^{2^m}$ over an even-dimensional sphere $S^{2m} \subset \mathbb{R}^{2m+1}$. Consider the section $\beta \in \Gamma\left(P^- M\right)$ defined by

$$\beta = p\left(E_1\right) p\left(E_2\right) \ldots p(E_{2m+1}),$$

where $\{E_1, \ldots, E_{2m+1}\}$ is the standard basis of vector fields in $\mathbb{R}^{2m+1}$, and where $p_x : T_x\mathbb{R}^{2m+1} \to T_x S^{2m}$ is the orthogonal projection. (Note that a similar bundle map may be constructed on any spin^c submanifold of $\mathbb{R}^{2m+1}$.) Observe that $\beta \in \Gamma\left(P^- M\right)$, and its zeros are the points of intersection of the axes in $\mathbb{R}^{2m+1}$ with S^{2m}. Note that these zeros are nondegenerate. To see this, renumber the axes so that the zero in question is the axis parallel to E_1. Near this zero, the vector fields $p\left(E_2\right), \ldots, p\left(E_{2m+1}\right)$ are nonzero, and $p\left(E_1\right) = \pm\left(\sin r\right)\partial_r$, where r is the geodesic radial coordinate. Since $p\left(E_1\right)$ is locally the gradient of $\mp\cos\left(r\right)$, which is a Morse function near $r = 0$, $p\left(E_1\right)$ has nondegenerate zeros. Also, note that the index of $p\left(E_1\right)$ is 1, since it is either a source or sink. Thus β has nondegenerate zeros. We now compute the index of β at each zero. At one of the zeros of $p\left(E_j\right)$, we write

$$\begin{aligned}\beta &= p\left(E_1\right) p\left(E_2\right) \ldots p\left(E_{2m+1}\right)\\ &= p\left(F_j\right) p\left(E_1\right) \ldots p\left(E_{j-1}\right) p\left(E_{j+1}\right) \ldots p\left(E_{2m+1}\right)\\ &= \pm p\left(F_j\right) p\left(\pm E_1\right) \ldots p\left(E_{j-1}\right) p\left(E_{j+1}\right) \ldots p(E_{2m+1}),\end{aligned}$$

where F_j is a vector field with two nondegenerate zeros (at the zeros of E_j) whose index is $(-1)^{j-1}$ at each of those zeros. We have used Lemma 6.8. The form of β above implies that the index of β at each of the two zeros of $p\left(E_j\right)$ is $(-1)^{j-1}$. This verifies Theorem 6.11, which implies that

$$\begin{aligned}\chi\left(S^{2m}\right) &= \sum_{\overline{x}} \mathrm{ind}_\beta\left(\overline{x}\right)\\ &= \sum_{j=1}^{2m+1}\left[(-1)^{j-1} + (-1)^{j-1}\right]\\ &= 2.\end{aligned}$$

6.3. The induced index on submanifolds

Let F be a real, oriented vector bundle over M with odd rank. Let $E = E^+ \oplus E^-$ be a graded, self-adjoint Clifford module over the total space of F. Identify the zero section of F with M; the inclusion $TM \subset TF$ then induces a graded $\mathbb{C}\mathrm{l}\left(TM\right)$-action on E over M.

Proposition 6.14. *The index of the Dirac operator associated to the* $\mathbb{C}\mathrm{l}\left(TM\right)$*-module E over M is zero.*

Proof. Let ω be a nonzero vertical volume form that induces the orientation of F. Then $Z = \mathrm{i}c(\omega) : E^\pm \to E^\mp$ is a nonsingular, self-adjoint bundle map that anticommutes with Clifford multiplication by sections of TM. By Remark 3.4, the index of the corresponding Dirac operator is zero. □

Clearly, the same result would apply to an oriented submanifold of odd codimension in a manifold endowed with a given Clifford module; the vector bundle F is the normal bundle of the submanifold. It would also apply to a component of the boundary of a manifold with boundary.

Remark 6.15. This resembles the cobordism invariance of the index of Dirac operators. See [11, Chapter XVII]. A perturbation proof of the standard cobordism invariance result is given by Braverman in [4,5]. In our case, the bundle map is an odd endomorphism, and we do not require that the manifold with boundary be compact.

Appendix A. Example when no localization occurs

In this section, we give an example of a perturbation of the Dirac operator that yields first order terms and where no localization occurs. This motivates the requirement that $ZD + DZ$ be a bundle map in Section 2.

Let $D_s : \Gamma\left(S^1, \mathbb{C}\right) \to \Gamma\left(S^1, \mathbb{C}\right)$ be defined by

$$D_s f := \mathrm{i}\frac{\mathrm{d}f}{\mathrm{d}\theta} + s\left(\sin\theta\right) f.$$

Then D_s is a family of essentially self-adjoint differential operators. We solve the equation $D_s f = \lambda f$ by separating variables. We conclude that the eigenvalues of D_s are $\lambda_{\pm n} = \pm n$, and the corresponding orthonormal eigenfunctions are

$$f_{\pm n}(\theta, s) = \frac{1}{\sqrt{2\pi}} \mathrm{e}^{\mathrm{i}(\mp n\theta - s\cos\theta)}.$$

Clearly, as $s \to \infty$, eigenvalues $\lambda_{\pm n}$ are fixed, and the eigenfunctions do not localize in the usual sense since the magnitude of each eigenfunction stays constant: $|f_n(\theta, s)| = \frac{1}{\sqrt{2\pi}}$.

In this case, $D = \mathrm{i}\frac{\mathrm{d}}{\mathrm{d}\theta}$, and $Z = \sin\theta$ can be thought of as Clifford action by the complex vector field $-\mathrm{i}\sin\theta\frac{\partial}{\partial\theta}$. The anticommutator of D and Z is

$$DZ + ZD = \mathrm{i}\left(2\cos\theta\frac{\mathrm{d}}{\mathrm{d}\theta} - \sin\theta\right),$$

and the operator

$$D_s^2 = -\frac{\mathrm{d}^2}{\mathrm{d}\theta^2} + s\left(2\mathrm{i}\cos\theta\frac{\mathrm{d}}{\mathrm{d}\theta} - \mathrm{i}\sin\theta\right) + s^2\cos^2\theta$$

contains a first order term, as expected.

Appendix B. Graded Clifford bundles and Dirac operators

We recall several well known facts about Clifford structures on manifolds of even and odd dimensions. The books [3,9,14] are standard references.

B.1. The Clifford bundle

As before, (M, g) is an oriented Riemannian manifold of $\dim M = n$. For any $x \in M$, we denote by $\mathrm{Cl}(T_xM)$ the Clifford algebra of the tangent space T_xM. The spaces T_xM and T_x^*M are canonically isomorphic, using the chosen Riemannian metric.

The Clifford algebra $\mathrm{Cl}(T_xM)$ is the direct sum of even and odd components, denoted $\mathrm{Cl}^+(T_xM)$ and $\mathrm{Cl}^-(T_xM)$.

The *Clifford bundle* $\mathrm{Cl}(TM)$ of M is the Z_2-graded bundle over M whose fiber at $x \in M$ is $\mathrm{Cl}(T_xM)$ ([3], 3.30). We will denote the *complexified Clifford algebra* $\mathrm{Cl}(T_xM) \otimes \mathbb{C}$ by $\mathbb{C}\mathrm{l}(T_xM)$ and the *complexified Clifford bundle* $\mathrm{Cl}(TM) \otimes \mathbb{C}$ by $\mathbb{C}\mathrm{l}(M)$.

The Levi-Civita connection ∇^{TM} induced by the Riemannian metric g extends canonically to a connection on $\mathbb{C}\mathrm{l}(T_xM)$ compatible with the grading and the Clifford multiplication.

B.2. Clifford modules

A *graded self-adjoint Clifford module* ([3], 3.32) on a manifold M is a Z_2-graded complex vector bundle $E = E^+ \oplus E^-$ together with a bundle endomorphism

$$c : TM \to \mathrm{End}(E).$$

such that the following properties hold: for any $x \in M$ and any vectors $v, w \in T_xM$

(i) $c(v) : E_x^{\pm} \to E_x^{\mp}$ is a graded action;

(ii) $c(v)c(w) + c(w)c(v) = -2g_x(v, w)\mathbf{1}$, where g_x is the metric on T_xM;

(iii) the bundle E is equipped with a Hermitian metric such that the subbundles E^+ and E^- are orthogonal and the operator $c(v)$ is skew-adjoint ;

(iv) E is equipped with a grading-preserving Hermitian connection $\nabla = \nabla^E$ satisfying

$$\left[\nabla_V^E, c(W)\right] = c\left(\nabla_V^{TM} W\right),$$

for arbitrary vector fields V and W on M. This connection is called a *Clifford connection* ([3], 3.39). Clifford connections always exist ([3], 3.41).

B.3. Twisted Clifford modules

Given a Clifford module E and a vector bundle F over M, we can construct the *twisted Clifford module* $E \otimes F$ *obtained from* E *by twisting with* F. The Clifford action on $E \otimes F$ is given by $c(v) \otimes 1$. Given a connection ∇^F on F we can define the *product connection* $\nabla^E \otimes 1 + 1 \otimes \nabla^F$ on $E \otimes F$.

B.4. The chirality operator and the induced grading on E

Let $e_1, \ldots, e_n$ be an oriented orthonormal basis of $T_x M$. We consider the element

$$\gamma = i^k c(e_1) \ldots c(e_n) \in \mathrm{End}(E_x),$$

where $k = n/2$ if n is even and $k = (n+1)/2$ if n is odd. This element is independent of the choice of basis and anticommutes with any $c(v)$ where $v \in T_x M$ if n is even and commutes if n is odd. Moreover, $\gamma^2 = \mathbf{1}$ ([3], 3.17). The *chirality operator* γ is the section of End (E) that restricts to the element above on each fiber. The bundle map γ has eigenvalues ± 1, and we can define subbundles

$$E_\gamma^\pm = \{\alpha \in E : \gamma\alpha = \pm\alpha\}.$$

The grading $E = E_\gamma^+ \oplus E_\gamma^-$ is called the *grading induced by* γ on E or the *natural grading* on E.

B.5. The Dirac operator

The *Dirac operator* $D : \Gamma(M, E) \to \Gamma(M, E)$ associated to a Clifford connection ∇^E is defined by the following composition

$$\Gamma(M, E) \xrightarrow{\nabla^E} \Gamma\left(M, T^*M \otimes E\right) \xrightarrow{c} \Gamma(M, E).$$

In local coordinates this operator may be written as

$$D = \sum_{i=1}^n c(\mathrm{d}x_i) \nabla_{\partial_i} : \Gamma\left(M, E^\pm\right) \longrightarrow \Gamma\left(M, E^\mp\right).$$

This is a first order elliptic operator. Moreover, it is formally self-adjoint and essentially self-adjoint with the initial domain smooth, compactly supported sections ([3], p. 119). Its *principal symbol* is given by

$$\sigma_D(x, \xi) = \mathrm{i}c(\xi) : \Gamma\left(M, T_x^* M\right) \to \mathrm{End}(E_x).$$

B.6. The spin and spin^c bundles

Let M be an even-dimensional oriented manifold with spin structure, and let $\mathbb{S} = \mathbb{S}^+ \oplus \mathbb{S}^-$ be a complex spinor bundle over M with the grading induced by γ. It is a minimal Clifford module; i.e. for any other Clifford module E over M there is a vector bundle F such that we have an isomorphism of Clifford modules

$$E \cong \mathbb{S} \otimes F,$$

where $F = \mathrm{Hom}_{\mathbb{C}\mathrm{l}(TM)}(\mathbb{S}, E)$ and the Clifford action is trivial on the second factor ([3], Sect. 3.3).

If the dimension of M is odd, then any Clifford module E over M is isomorphic to

$$E \cong (\mathbb{S} \otimes F_1) \oplus (\mathbb{S} \otimes F_2)$$

where $v \in \Gamma(M, TM)$ acts on $\mathbb{S} \otimes F_1$ by $c(v) \otimes \mathbf{1}$ and on $\mathbb{S} \otimes F_2$ by $c(-v) \otimes \mathbf{1}$. In the odd case we denote the first action $c^+(v) \otimes \mathbf{1}$ and the second action $c^-(v) \otimes \mathbf{1}$.

A connection ∇^E on the twisted Clifford module $E = \mathbb{S} \otimes F$ is a Clifford connection if and only if

$$\nabla^E = \nabla^{\mathbb{S}} \otimes 1 + 1 \otimes \nabla^F$$

for some connection ∇^F on F.

Note that there are global obstructions to the existence of complex spinor bundles (see [3], 3.34); however, locally the decompositions above always exist.

Every spin manifold and every almost complex manifold has a canonical spin^c structure. In addition, every oriented, compact manifold of dimension ≤ 3 is spin^c [10].

B.7. Local classification of gradings

The above results apply to Dirac operators over bundles with the natural grading — that induced directly from the grading on complex spinors. The following lemma classifies all possible gradings for Clifford representations (and thus Dirac operators).

Let V be an oriented, real Euclidean vector space, and let $E = E^+ \oplus E^-$ be a complex vector space that is a graded $\mathbb{C}\mathrm{l}(V)$-module. Let $\mathbb{S}$ denote the irreducible representation space of $\mathbb{C}\mathrm{l}(V)$. Let $c(v)$ denote the Clifford multiplication by $v \in V$ on $\mathbb{S} = \mathbb{S}^+ \oplus \mathbb{S}^-$ when V is even-dimensional, and let $c^+(v) : \mathbb{S} \to \mathbb{S}$ and $c^-(v) := -c^+(v) : \mathbb{S} \to \mathbb{S}$ be the Clifford multiplications that generate the two nonequivalent irreducible representations of $\mathbb{C}\mathrm{l}(V)$ when V is odd-dimensional. We let

$$c^E(v) : E^\pm \to E^\mp$$

denote the Clifford action by v on the graded vector space E.

Lemma B.1. *With the notation described above, there exists a complex vector space W such that*

(1) $E \cong \mathbb{S} \otimes W$, *where* $W \cong \mathrm{Hom}_{\mathbb{C}\mathrm{l}(V)}(\mathbb{S}, E)$.
(2) *If the dimension of V is even, then the associated Clifford action on $\mathbb{S} \otimes W$ is $c(v) \otimes \mathbf{1}$. In addition, $W = W^+ \oplus W^-$ for some orthogonal subspaces $W^\pm$ of W, and $E^\pm \cong (\mathbb{S}^+ \otimes W^\pm) \oplus (\mathbb{S}^- \otimes W^\mp)$.*
(3) *If the dimension of V is odd, then there exists an orthogonal decomposition $W = W' \oplus W'$, such that*

$$E \cong \mathbb{S} \otimes W \cong (\mathbb{S} \otimes W') \oplus (\mathbb{S} \otimes W'),$$

the induced action of $c^E(v)$ on $(\mathbb{S} \otimes W') \oplus (\mathbb{S} \otimes W')$ is given by

$$\left(c^+(v) \otimes \mathbf{1}, c^-(v) \otimes \mathbf{1}\right) = \left(c^+(v) \otimes \mathbf{1}, c^+(v) \otimes -\mathbf{1}\right),$$

and

$$E^\pm \cong \mathbb{S} \otimes \mathrm{span}\left\{(w, \pm w) \in W' \oplus W' \mid w \in W'\right\}.$$

Proof. The first statement follows directly from the general facts about the representation theory of Clifford algebras; see Appendix B.6.

To prove (2), observe that the even part of the Clifford algebra $\mathbb{C}\mathrm{l}^+(V)$ acts by endomorphisms on E^+ and E^-. This leads to a representation of spin$^c(n) \subset \mathbb{C}\mathrm{l}^+$ on E^+ and E^-. There are exactly two nonequivalent irreducible representations of spin$^c(n)$, given by the actions of $\mathbb{C}\mathrm{l}^+$ on $\mathbb{S}^+$ and $\mathbb{S}^-$; see [10, p. 432]. Thus, there are complex vector spaces W^+ and W^- such that $E^+ \cong (\mathbb{S}^+ \otimes W^+) \oplus (\mathbb{S}^- \otimes W^-)$, which implies that $E^- \cong (\mathbb{S}^- \otimes W^+) \oplus (\mathbb{S}^+ \otimes W^-)$.

To prove (3), where V is odd-dimensional, observe that there is a unique irreducible representation of spin$^c(n)$, given by the action of $\mathbb{C}\mathrm{l}^+$ on $\mathbb{S}$. We have $E = E^+ \oplus E^- \cong (\mathbb{S} \otimes W^+) \oplus (\mathbb{S} \otimes W^-) \cong \mathbb{S} \otimes (W^+ \oplus W^-)$ with the Clifford action on the last term being $c^E(v) = c^+(v) \otimes J$. Here $J : W^\pm \to W^\mp$. The operator J must be Hermitian and squares to identity (since $c^E(v)$ is skew-Hermitian and squares to $-\mathbf{1}$).

Now let $\{e_1, \ldots, e_k\}$ be an orthonormal basis of W^+; then $\{Je_1, \ldots, Je_k\}$ must be an orthonormal basis of W^-. Thus W has an orthonormal basis $\{e_1, \ldots, e_k, Je_1, \ldots, Je_k\}$. We introduce a new decomposition of $W = W_1 \oplus W_2$, where $W_1 = \mathrm{span}\left\{e_1 + Je_{1,\ldots,}e_k + Je_k\right\}$ and $W_2 = \mathrm{span}\{e_1 - Je_1, \ldots, e_k - Je_k\}$. Then J is the identity on W_1 and minus the identity on W_2, and we can decompose E as

$$E \cong (\mathbb{S} \otimes W_1) \oplus (\mathbb{S} \otimes W_2),$$

where $c^E(v)$ acts by $\left(c^+(v) \otimes \mathbf{1}, c^+(v) \otimes -\mathbf{1}\right)$. The conclusion (3) follows from the observation that $W' := W_1 \cong W_2$, where the isomorphism maps each $e_m + Je_m$ to $e_m - Je_m$. □

B.8. Global classification of gradings

Let $E = E^+ \oplus E^-$ be a graded, self-adjoint, Hermitian $\mathbb{C}\mathrm{l}\,(TM)$-module over M. Suppose that M is spin^c. Choose a spin^c structure on M. Let $\mathbb{S}$ be the corresponding spin^c bundle over M, so that the representation of $\mathbb{C}\mathrm{l}\,(TM)$ is irreducible on $\mathbb{S}$ and $\mathbb{C}\mathrm{l}\,(TM) \cong \mathrm{End}\,(\mathbb{S})$. Let $c\,(v)$ denote the Clifford multiplication by $v \in TM \otimes \mathbb{C}$ on $\mathbb{S}$; $\mathbb{S} = \mathbb{S}^+ \oplus \mathbb{S}^-$ if M is even-dimensional. Let $c^+\,(v) : \mathbb{S} \to \mathbb{S}$ and $c^-\,(v) := -c^+\,(v) : \mathbb{S} \to \mathbb{S}$ be the Clifford multiplications that generate the two irreducible representations of $\mathbb{C}\mathrm{l}\,(TM)$ when M is odd-dimensional. We let

$$c^E\,(v) : E^\pm \to E^\mp$$

denote the Clifford multiplication by v on the graded vector bundle E.

Corollary B.2 (*Classification of Bundles of Graded Clifford Modules*). *If M is* spin^c *with the above notation, there exists a complex vector bundle W such that*

(1) $E \cong \mathbb{S} \otimes W$, *where* $W \cong \mathrm{Hom}_{\mathbb{C}\mathrm{l}(M)}\,(\mathbb{S}, E)$.
(2) *When M is even-dimensional, the associated Clifford action on $\mathbb{S} \otimes W$ is $c\,(v) \otimes \mathbf{1}$. Then $W = W^+ \oplus W^-$ for some orthogonal vector subbundles $W^\pm$ of W, and $E^\pm \cong \left(\mathbb{S}^+ \otimes W^\pm\right) \oplus \left(\mathbb{S}^- \otimes W^\mp\right)$.*
(3) *When M is odd-dimensional, there exists an orthogonal decomposition $W = W' \oplus W'$, such that*

$$E \cong \mathbb{S} \otimes W \cong \left(\mathbb{S} \otimes W'\right) \oplus \left(\mathbb{S} \otimes W'\right),$$

the induced action of $c^E\,(v)$ on $\left(\mathbb{S} \otimes W'\right) \oplus \left(\mathbb{S} \otimes W'\right)$ is given by

$$\left(c^+\,(v) \otimes \mathbf{1}, c^-\,(v) \otimes \mathbf{1}\right) = \left(c^+\,(v) \otimes \mathbf{1}, c^+\,(v) \otimes -\mathbf{1}\right),$$

and

$$E^\pm \cong \mathbb{S} \otimes \mathrm{span}\left\{(w, \pm w) \in W' \oplus W' \mid w \in W'\right\}.$$

Finally, if M is not spin^c, *then the relevant facts above are true locally but not globally; that is, the bundles $\mathbb{S}$, W, $W^\pm$, W' can be defined on a sufficiently small neighborhood of any given point of M, and the properties above are true over that neighborhood, but $\mathbb{S}$ cannot be extended to a globally defined* spin^c *bundle.*

Proof. The fact that $E \cong \mathbb{S} \otimes W$ in both the odd and even cases follows by setting $W = \mathrm{Hom}_{\mathbb{C}\mathrm{l}}\,(\mathbb{S}, E)$, the bundle maps from $\mathbb{S}$ to E that are $\mathbb{C}\mathrm{l}\,(TM)$-equivariant. (In the odd case, one must fix an irreducible representation c^+ on $\mathbb{S}$ once and for all.) The isomorphism $\mathbb{S} \otimes W \to E$ is given by $s \otimes w \mapsto w\,(s)$.

In the even case, observe that the bundles $W^\pm$ may be defined globally by noting that for example $\left(\mathbb{S}^+ \otimes W^\pm\right) = E^\pm \cap \left(\mathbb{S}^+ \otimes W\right)$, where we have abused notation using the isomorphism $\mathbb{S} \otimes W \to E$.

In the odd case, observe that in the proof of Lemma B.1, the representation theory alone determines the bundles W_1 and W_2 from W, and the construction of W' is canonical. The result follows. □

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